University Studies
Second Series—No. I

ANALYTICAL GEOMETRY
OF
HYPER-SPACES

THESIS PRESENTED TO THE UNIVERSITY OF CALCUTTA
FOR THE PREMCHAND ROYCHAND STUDENTSHIP
EXAMINATION OF 1914.

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PUBLISHED BY THE
UNIVERSITY OF CALCUTTA
1918

Printed by Atulchandra Bhattacharyya, at the
Calcutta University Press, Senate House, Calcutta.

PREFACE

The present Thesis deals with certain interesting problems in n-dimensional Geometry. The method adopted is one of deduction from first principles. Although the physical existence of spaces of dimensions higher than three may be controversial, the theoretical investigation of the properties of such spaces is quite logical. The subject lends itself easily to analytical treatment, combined with occasional use of geometrical methods. The first ideas were taken from a paper published by Dr. S. D. Mookerjee, M.A., Ph.D., in the Bulletin of the Calcutta Mathematical Society. In the absence of any regular treatises on the subject in the English language, I had to work out all the results independently and consequently most of the theorems in the present thesis are the results of my independent thinking.

Most of the original workers in Higher Geometry are either Italian, German or French. I had to go through some standard works in those languages in order to be able to compare my results with those of original authors. The mode of treatment in these works is found to be different from that of mine. These authors have confined themselves chiefly to the study of construction of Hyper-spaces, with special reference to their projective properties, while my attention has been devoted mostly to metric properties. In fact, my attempts have been confined to a systematic and legitimate extension of two or three dimensional Euclidean Geometries, and the present Thesis might reasonably be termed as " General Euclidian Geometry." " The science of Abstract Geometry presents itself in two ways :—as a legitimate extension of ordinary two and three dimensional Geometries and as a need in these Geometries and in analysis generally. Whenever we are concerned with quantities connected together in any manner, and which are, or are considered, as variable or determinable, then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them (if only two or three in number) as the co-ordinates of a point in a plane or space; for more than three quantities the case becomes more complex and there is greater need of such representation. But this can only be obtained by means of the notion of a space of the proper dimensionality."*

The notion of Hyper-spaces is spontaneously presented to Mathematicians, who, having to deal with questions on more than three variables, interpret this fact in words analogous to those in ordinary analytical Geometries of two or three dimensions and seek a solution in this new light. In order to trace the origin of this notion we must refer to former writers.† An explicit exposition of it in various directions commences from the middle of the last century, in the works of Cayley, Grassmann, Riemann, Beltrami, Betti, Lie, Klein, Schläfli, Jordan, Clifford, &c. Although these authors have enunciated many ideas and theorems on the Geometry of Hyper-spaces, nevertheless it may be said that the first original work of general character is due to Veronese.‡ In his great work Veronese has exposed a system of postulates relating to the metric and projective Geometry of spaces of any number of dimensions. This work has been translated into the German language by Adolf Schepp, and contains a synthetic treatment not only of Euclidian spaces but also of Riemann's and Lobatschewsky's as well.

The number of books and memoirs on the Geometry of higher demensions has increased enormously in recent years. In 1870 Cayley published his " Memoir on Abstract Geometry. Jordan, in 1875, gave a methodical generation of metrical geometry by means of Cartesian coordinates. In 1882 Veronese published his great work. Schoute,* in a book of two volumes, made the subject very clear and interesting by using a number of methods. A remarkable memoir on geometry of n dimensions was written by L. Schlafli† in 1850-1852, but was published after his death in 1911. He works out in great details the theory of perpendicularity and the angle-concepts. As I had no occasion to consult most of these authors I cannot assert with certainty how far my attempts have been successful in producing new results, or how far I might have been anticipated by them.

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* *Vide* A. Cayley,—Phil. Trans. of the Royal Soc. of London, Vol. CIX, 1870.

† Segre—Su alcuni indirizzi nelle investigazioni geometriche (Rivista di Matematica 1891). Translated into English in the Bulletin of the American Math. Soc. 10 (2), 1904,

Jordan,—Essai sur la géométrie à n dimensions—Bull. de la Soc. Math. de France, Vol. III.

‡ Grundzüge der Geometrie von mehreren Dimensionem und Arten etc.

In Chapter I, Hyper-spaces have been represented by means of analytical equations and in some places proper interpretations to these equations have been given. At the outset I have started with the notion of n co-ordinates of a point; but at a later stage several postulates enunciated by Prof. Cayley have been assumed.

In Chapter II, I have dealt with angles between spaces. The angle-concepts between spaces were first studied algebraically by Jordan (1875), geometrically by Cassani‡ (1885) and by Castelnuovo§ (1885). Castelnuovo in the course of his investigations proved a very elegant theorem that the r angles between two r-spaces S_r and S'_r in a $2r$-space are all real. Veronese has given the following definition of angles between two Spaces :—||

"Indem wir eine ähnliche Methode befolgen wie bie der Definition der Winkel von Strahlen, Halbebenen und Halbräumen von drei Dimensionen (Theil I, Buch III, Kap 1, 7. und Theil II, Buch 1, Kap 1, 7), verstehen wir unter den Winkeln

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* *Schoute*,—Mehrdimensionale Geometrie, Sammlung Schubert, XXXV and XXXVI. (Leipzig.)

† *Schlafli*—Theorie der vielfachen Kontinuität.

‡ *Cassani*—Rendiconti dell' Acc dei Lincei, (1885.)

§ *Castelnuovo*—Atti del R. Istituto Veneto, (1885.)

|| Veronese—Grundzüge der Geometrie &c. §. 169, Bem II.

zweier Räume S_r and S_m diejenigen, welche durch die kleinsten normalen Abstände ihrer unendlich fernen Räume gemessen werden."

In Chapter III, some properties of Hyper-Spheres have been considered.

I must refer to the fact that at the time of preparing the thesis, I had no opportunity of consulting the work "Matrices and Determinoids" of Dr. C. E. Cullis. But I hope that most of my results could be simplified by introducing the elegant notations used in that book.

In conclusion, I beg leave to acknowledge my indebtedness to the Hon'ble Justice Sir Asutosh Mookerjee Kt., C.S.I., M.A., D.L., etc. who encouraged the prosecution of my studies in Hyper-Geometry, to Dr. Shyamadas Mookerjee, M.A., Ph.D. for help with valuable suggestions in preparing the thesis and to the authorities of the Calcutta University for its publication.

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October, 1914.

S. M. GANGULI.

Symbols and Abbreviations used.

(1) The line 'l'—stands for the half-line whose direction-cosines are $(l_1, l_2, l_3, \ldots l_n)$.

(2) The plane (l, m)—stands for the plane determined by the two intersecting lines 'l' and 'm.'

(3) The angle $\widehat{lm}$—stands for the plane angle between the lines 'l' and 'm'.

(4) The symbol (lm)—stands for "Cosine of the angle $\widehat{lm}$."

(5) The ,, $[lm]$— ,, ,, "Sine of the angle $\widehat{lm}$."

(6) ,, ,, $[lm/pq]$—stands for the function

$$\sum \begin{vmatrix} l_r & l_s \\ m_r & m_s \end{vmatrix} \begin{vmatrix} p_r & p_s \\ q_r & q_s \end{vmatrix}$$

$[r=1,2,3,\ldots n;\ s=r+1, r+2,\ldots n]$

etc. etc.

ANALYTICAL GEOMETRY

OF

HYPER-SPACES.

CHAPTER I—General Properties.

§ 1. **The Euclidean Hyper-space of n-dimensions :—** The Euclidean Hyper-space of n-dimensions (or an n-dimensional space or simply an n-*space*) is characterised by the fundamental property that through any point of it can be drawn n right lines, and only n, which are mutually at right angles.

§ 2. For purposes of analytical investigations in the Hyper-space of n-dimensions, we refer all our elements to n axes, mutually orthogonal, drawn through any point as origin. The coordinates of any point are determined as follows :—

Let P be the point and O the origin. Then, if $M_1, M_2, M_3 \ldots\ldots M_n$ are the projections of P on the n axes through O, OM_1, OM_2, $\ldots\ldots OM_n$ are the n coordinates of P and are generally denoted by $x_1, x_2, \ldots\ldots x_n$. This is an extension of ordinary Geometries of two or three dimensions ; but we may start from a different point of view with only n unknowns, and conceive relation or relations between them. Professor Eugenio Bertini of Pisa starts with this latter view, as is evident from the following lines :—

"Astraendo da ogni possibile rappresentazione geometrica od intuitiva, si definisce Spazio ad r dimensioni e si indica con S_r, o anche con $[r]$, la totalità formata da tutti i gruppi di valori (reali o complessi) dei rapporti di $(r+1)$ parametri $x_0, x_1, \ldots\ldots x_r$: escluso che questi parametri possano diventare tutti nulli o alcuno infinito. Ogni tale gruppo dicesi punto dell' S_r e le x_0, $x_1, \ldots\ldots x_r$ si chiamano le sue coordinate omogenee, mentre si dicono coordinate non omogenee del punto i raporti di r delle $x_0, x_1, \ldots\ldots x_r$ alla rimanente."*

§ **3**. If a chain of lines $0\ 0_1\ 0_2\ 0_3\ldots\ldots 0_n$ be drawn from the origin and if the successive links of this chain be equal and parallel to $x_1, x_2, x_3\ldots\ldots x_n$ respectively, 0_n must coincide with P.

For the projection of $0P$ on the rth axis is x_r and the projection of this chain on the same axis is also x_r. Therefore the projection of $P0_n$, which is the difference of these two projections, is zero on the rth axis. But this rth axis may be any one of the n axes. Hence $P0_n$ must either vanish or be perpendicular to each of the n axes. But in an Hyper-space of n dimensions there cannot be more than n axes through a point, mutually orthogonal. Hence P must coincide with 0_n.†

§ **4**. Hence follows that, if r is the length of the line $0P$ and l_i $(i=1, 2, 3,\ldots\ldots n)$ its direction-cosines,

$$r^2 = x_1^2 + x_2^2 + x_3^2 + \ldots\ldots + x_n^2$$

$$1 = l_1^2 + l_2^2 + l_3^2 + \ldots\ldots + l_n^2.$$

Also,

$$rr^1 \cos\theta = x_1 x_1^1 + x_2 x_2^1 + x_3 x_3^1 + \ldots\ldots + x_n x_n^1,$$

or,

$$\cos\theta = l_1 l_1^1 + l_2 l_2^1 + l_3 l^1{}_3 + \ldots\ldots + l_n l_n^1,$$

where $(x_1^1, x_2^1,\ldots\ldots x_n^1)$ are the coordinates of another point P^1, r^1 the length OP^1 and $(l_1^1, l_2^1\ldots\ldots l_n^1)$ its direction-cosines and θ the angle between OP and OP^1.

§ **5**. **Hyper-space of r-dimensions contained in an Euclidean n-space:**—An r-space may be defined as a space determined as the join of $(r+1)$ given points $(r<n)$, provided *no* two groups of points, taken from the given points, are "*conjoint*" *i.e.* determine identical dimensionals as their joins.

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* *Bertini.*—Introduzione alla Geometria Proiettiva degli Iperspazi, Cap°I.

English translation.—From all possible geometric or intuitive representations, a space of r-dimensions denoted by S_r or by $[r]$, is defined as the aggregate of the groups of values (real or complex) of the ratios of $(r+1)$ parameters $x_0, x_1, x_2 \ldots . x_r$, where none of these parameters is infinite and all are different from zero. Every such group is called a point of S_r and $x_0, x_1, x_2\ldots\ldots x_r$ are called its "Homogeneous coordinates," while the ratios of r of $x_0, x_1\ldots\ldots x_r$ to the remaining one are called the "non-homogenous coordinates."

† This proof has been taken from Dr. S. D. Mookerjea's Paper I on "Parametric coefficients &c."—Bulletin Calcutta Math. Soc. Vol. I, No. 3, 1909.

This join of $(r+1)$ points generally means the "indefinite space" through the given points.*

Prof. E. Bertini gives the following definition of a k-space in an r-space :—

Let x_0, x_1, x_2,......x_k, be any $(k+1)$ independent points in an r-space. Then the coordinates of any point x in the k-space determined by these points are given by

$$\text{(A)}\quad x_i = \lambda_0 x_{0i} + \lambda_1 x_{1i} + \ldots\ldots \lambda_k x_{ki} \qquad (i=0, 1, 2, 3, \ldots\ldots r)$$

where $\lambda_0 + \lambda_1 + \ldots\ldots + \lambda_k = 1$

All the points, which are obtained from (A) by varying the parameters λ_0, λ_1,......λ_k, are said to constitute an Aggregate (∞^k): and as the points of this aggregate may be made to correspond uniquely to the system of values of the parameters, the same Aggregate is a "*Spazio lineare* (*or simply Spazio*) *a k dimensioni*" and exactly a "*Spazio* S_k *subordinato dell'* S_r."†

The definition given by Veronese is essentially the same as we have given above, as will appear from the following lines:— "$(n+1)$ *unabhängige punkte bestimmen einen Raum von n Dimensionen und dieser wird durch* $(n+1)$ *seiner unabhängigen punkte bestimmt.*" ‡

§ 6. To find the condition that (r+1) given points in an n-space may not be "conjoint," or that the (r+1) points may be "independent."

Let x_{ji} $(j=0, 1, 2, \ldots\ldots r;\ i=1, 2, \ldots\ldots n)$ be the coordinates of $(r+1)$ given points.

Now consider the following $(n+1)$ linear equations :—

$$\text{(I)}\quad \ldots\left\{\begin{array}{lllll}
\lambda_0 & +\lambda_1 & +\lambda_2 + \ldots\ldots & +\lambda_r & =0 \\
\lambda_0 x_{01} & +\lambda_1 x_{11} & +\ldots\ldots\ldots\ldots & +\lambda_r x_{r1} & =0 \\
\lambda_0 x_{02} & +\lambda_1 x_{12} & +\ \ldots\ldots\ldots & +\lambda_r x_{r2} & =0 \\
\ldots & \ldots & \ldots\quad \ldots & \ldots & \\
\lambda_0 x_{0n} & +\lambda_1 x_{1n} & +\ \ldots\ldots\ldots & +\lambda_r x_{rn} & =0
\end{array}\right.$$

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* *Vide.*—Bulletin Calcutta Math. Soc. Vol. I, No. 3, 1909.

† *Bertini.*—Ibid, Cap°I, *n* 5.

‡ *Veronese.*—" Fondamenti di Geometria &c.," translated into German by Adolf Schepp,—" Grundzüge der Geometrie von mehreren Dimensionem, &c." —§ 157, Satz. III, Zus.

English translation—$(n+1)$ "independent" points determine a space of n dimensions and a space of n-dimensions is determined by $(n+1)$ of its "independent" points..

which are obtained by equating to zero the same linear combinations of the coordinates of the points with the parameters λ_0, $\lambda_1, \ldots\ldots \lambda_r$, subject to the condition $\lambda_0 + \lambda_1 + \ldots\ldots + \lambda_r = 0$. These $(r+1)$ points will be *conjoint* or not (dependent or independent), according as there are or are not values of the parameters λ_0, λ_1, $\lambda_2, \ldots\ldots \lambda_r$, not all zeros, which satisfy (I), that is to say, (I) are satisfied only by values, not all zero, of the same parameters.

If $r+1 > n+1$, the points are necessarily "dependent" or "conjoint," because denoting by c $(\leq n+1)$ the characteristic of the matrix (I), they may be satisfied by taking at pleasure certain $r+1-c(>0)$ of the parameters λ_0, $\lambda_1, \ldots\ldots \lambda_r$.*

If $r+1 \leq n+1$, the characteristic c of the matrix (I) is $\leq r+1$. When $c = r+1$, the $(r+1)$ points are "independent," because (I) cannot be satisfied except by values, all zero, of λ_0, $\lambda_1, \ldots\ldots \lambda_r$.

If $c \angle r+1$, the $(r+1)$ points are "conjoint." With the convention that a matrix will be zero, when all its minors of maximum order are zero, and it will be different from zero in other cases,† it may thus be said that "$(r+1)$ *points are dependent or independent according as the matrix*

$$\begin{vmatrix} 1 & 1 & 1 \ldots\ldots\ldots 1 \\ x_{01} & x_{11} & x_{21} \ldots\ldots x_{r1} \\ \ldots & \ldots & \ldots \quad\quad \ldots \\ x_{0n} & x_{1n} & x_{2n} \ldots\ldots x_{rn} \end{vmatrix} = 0 \text{ or } \neq 0."$$

§ 7. The most general equation of the first order in an n-space may be taken as $a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n + a_{n+1} = 0$; and hence contains $(n+1)$ arbitrary constants. But we may divide out the whole expression by any one of the constants. Therefore the number of disposable constants is n, and the equation can be made to satisfy n conditions and no more:

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* *Capelli.*—Istituzioni di analisi algebrica, third edition, n, 445.

† *C E Cullis*—Matrices and Determinoids, Th. I., § 30.

e.g. it may be made to pass through n independent points. Thus any n independent points will determine uniquely the coefficients of an equation of the first degree and hence by § **5**, an equation of the first degree always represents an $(n\text{-}1)$-space or $(n\text{-}1)$-omal *.

N.B.—If any n points satisfy the above linear equation, any other point whose coordinates are of the form $\lambda_0 x_{0i} + \lambda_1 x_{1i} + \dots + \lambda_{n-1} x_{n-1\,i}$, will also satisfy the above equation, where $\lambda_0 + \lambda_1 + \dots\dots + \lambda_{n-1} = 1$; *i.e.*, the coordinates of any point in the $(n\text{-}1)$-space are of the form $\lambda_0 x_{0i} + \lambda_1 x_{1i} + \dots\dots + \lambda_{n-1} x_{n-1\,i}$. $[i=1, 2, 3, \dots n]$

§ 8. To find the equation of an Hyper-space of r dimensions passing through (r+1) given points ($r \angle n$).

Let the coordinates of the $(r+1)$ given points be denoted by x_{ji} $(j=0, 1, 2, 3, \dots\dots r\,;\; i = 1, 2, 3, \dots\dots n)$.

We have seen in § 7 that the coordinates x_i of any point x lying in the space determined by the given $(r+1)$ points may be written as $x_i = \lambda_0 x_{0i} + \lambda_1 x_{1i} + \dots\dots\dots\dots + \lambda_r x_{ri}$ $(i = 1, 2, 3, \dots\dots n)$ with the condition,

$$1 = \lambda_0 + \lambda_1 + \lambda_2 + \dots\dots\dots + \lambda_r .$$

Hence, since $r + 1 \angle n+1$, it is possible to find values of the parameters $\lambda_0, \lambda_1, \dots\dots \lambda_r$, not all zero, such that the above equations are satisfied and thus the locus of x_i $(i = 1, 2, 3, \dots n)$ is obtained by eliminating $\lambda_0, \lambda_1 \dots\dots\dots \lambda_r$ between the above equations :—

$$\begin{Vmatrix} x_1 & x_2 & x_3 & \dots\dots\dots & x_n \\ x_{01} & x_{02} & x_{03} & \dots\dots\dots & x_{0n} \\ x_{11} & x_{12} & x_{13} & \dots\dots\dots & x_{1n} \\ \dots & & \dots & & \dots \\ x_{r1} & x_{r2} & x_{r3} & \dots\dots\dots & x_{rn} \end{Vmatrix} = 0 \dots\dots\dots\dots\dots\dots (1)$$

This gives $(n\text{-}r)$ independent equations of the first order, Hence the intersection of $(n\text{-}r)$ spaces of $(n\text{-}1)$ dimensions is a space of r dimensions.

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* Cayley.

This exactly corresponds to what Prof. Cayley says that an (n-r)-fold linear relation determines an r-omal.*

§ 9. An r-space may be represented by means of analytical equations when one point and r independent lines through the point, all lying in the r-space, are given.

Let P (a_1, a_2, ... a_n) be the given point and let $l_{j\,i}$ ($j = 1, 2, 3, \ldots\ldots r$; $i = 1, 2, 3, \ldots\ldots n$) be the direction-cosines of the given lines through P.

Then any point on any of these lines must have its coordinates given by $a_i + r_j\ l_{j\,i}$, ($i = 1, 2, \ldots\ldots r$; $i = 1, 2, \ldots\ldots n$) where r_j is the distance of the point from P.

Thus altogether we have ($r + 1$) points of the r-space and its equations by § **8** can be written as :—

$$\begin{Vmatrix} x_1 & x_2 & \ldots\ldots\ldots & x_n & 1 \\ a_1 & a_2 & \ldots\ldots\ldots & a_n & 1 \\ a_1 + r_1 l_{11} & a_2 + r_1 l_{12} & \ldots\ldots\ldots & a_n + r_1 l_{1n} & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 + r_r l_{r1} & a_2 + r_r l_{r2} & \ldots\ldots\ldots & a_n + r_r l_{rn} & 1 \end{Vmatrix} = 0$$

which simplified give the following equations for the r-space :—

$$\begin{Vmatrix} x_1 - a_1 & x_2 - a_2 & \ldots\ldots\ldots & x_n - a_n \\ l_{11} & l_{12} & \ldots\ldots\ldots & l_{1n} \\ l_{21} & l_{22} & \ldots\ldots\ldots & l_{2n} \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ l_{r1} & l_{r2} & \ldots\ldots\ldots & l_{rn} \end{Vmatrix} = 0$$

which give (n-r) independent linear equations to determine the r-space.

§ **10**. We have defined that through any point of an n-space, n right lines, and only n, can be drawn mutually at right angles.

Of these, if the point lies in an r-space, r can be taken to lie in the r-space and hence the remaining (n-r) mutually orthogonal lines are such that they are orthogonal to r-lines in the r-space and therefore they are perpendicular to any line in the r-space drawn through the point.* These (n-r) orthogonal lines are therefore normal to the r-space.

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* **Cayley—A Memoir on Abstract Geometry, Phil. Trans. of the Royal Soc. of London, Vol. CLX, 1870.**

Hence we infer that through any point of an r-space, (n-r) independent normals to the r-space can be drawn. The (n-r) orthogonal lines determine an (n-r)-space orthogonal to the given r-space and all lines drawn through the point normal to the given r-space will be in this "normal space". †

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* This may be proved as follows :—

Let $l_{j\,i}$ ($i = 1, 2, 3, \quad n$; $j = 1, 2, 3, \ . \quad n$) be the direction-cosines of n mutually orthogonal lines, of which the first r may be taken to lie in the r-space.

The direction-cosines of any line in the r-space may be taken as—

$\sum\limits_{k=1}^{k=r} \lambda_k \, l_{k\,i}$ ($i = 1, 2, 3, \ \ldots \ n$), when λ_k is an arbitrary multiplier.

It will be sufficient to prove that any one of the remaining (n-r) lines,—the ($r+1$)th line for example,—is perpendicular to this line.

The condition of perpendicularity requires that

$$\sum_{i=1}^{i=n} l_{r+1\, i} \sum_{k=1}^{k=r} \lambda_k \, l_{k\,i} = 0$$

$$\text{or that,} \quad \sum_{k=1}^{k=r} \lambda_k \sum_{i=1}^{i=n} l_{r+1\, i} \, l_{k\,i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

But the ($r+1$)th line is orthogonal to the r lines and ∴ we have

$$\sum_{i=1}^{i=n} l_{k\,i} \, l_{r+1\, i} = 0, \quad (k = 1, 2, 3, \ \ldots\ldots \ r)$$

i.e. the coefficient of λ_k ($k = 1, 2, 3, \ \ldots\ldots\ldots \ r$) in (1) vanishes and hence the conclusion.

† The following proof may be added :—

The direction-cosines of any line must be of the form

$\sum\limits_{t=1}^{t=n} \lambda_t \, l_{t\,i}$, ($i = 1, 2, 3, \ . \ ..n$). If this is perpendicular to the r lines we must have

$$\sum_{i=1}^{i=n} l_{j\,i} \sum_{t=1}^{t=n} \lambda_t \, l_{t\,i} = 0, \ (j = 1, 2, 3, \ \ldots\ldots\ldots\ldots \ r)$$

$$\text{or} \quad \sum_{t=1}^{t=n} \lambda_t \sum_{i=1}^{i=n} l_{j\,i} \, l_{t\,i} = 0, \ (j = 1, 2, 3, \ \ldots\ldots\ldots \ r) \ \ldots\ldots\ldots \ (2)$$

In virtue of the relation that the n lines are mutually orthogonal we obtain from the r equations (2)

$$\lambda_1 = \lambda_2 = \lambda_3 = \ \ldots \ \ldots. = \lambda_r = 0,$$

and therefore, the direction-cosines of the line may be expressed as

$\sum\limits_{t=r+1}^{t=n} \lambda_t \, l_{t\,i}$, ($i = 1, 2, 3, \ \ldots\ldots\ldots \ n$) and this shews that the line lies in the normal (n-r)-space.

We can thus find the equations of an r-space, when one point in it and the direction-cosines of any set of (n-r) independent normals through the point are given.

For, let P $(a_1, a_2, \dots a_n)$ be the given point of an r-space and let the direction-cosines of (n-r) independent normals through P be given by $l_{j\,i}$ $(j = 1, 2, 3, \cdots n\text{-}r\,;\ i = 1, 2, 3, \dots n)$.

Let Q $(x_1, x_2, x_3, \dots x_n)$ be any point in the r-space. Then the direction - cosines of the line P Q are proportional to $(x_1 — a_1)$, $(x_2 — a_2)$ ....$(x_n ... a_n)$.

Thus, from the condition of perpendicularity of the line P Q and each of the (n-r) normals, we get (n-r) equations of the type $(x_1 - a_1)\, l_{j\,1} + (x_2 — a_2)\, l_{j\,2} +$ .... $+$ $(x_n — a_n)$ $l_{j\,n} = 0$ $(j=1, 2, 3,$ .... $n\text{-}r)$.

Therefore we obtain (n-r) *independent* linear equations to determine the r-space.

Cor :—From this we conclude that, if (n-r) independent linear equations are given to determine an r-space, the coefficients in each equation are proportional to the direction -cosines of a normal to the r-space.

Let us take the case of a right line in 3-dimensional geometry given by the equations $\begin{cases} a_1\, x + b_1\, y + c_1\, z + d_1 = 0 \\ a_2\, x + b_2\, y + c_2\, z + d_2 = 0 \end{cases}$

We know that (a_1, b_1, c_1) and (a_2, b_2, c_2) are respectively proportional to the direction-cosines of the normals to the two planes and hence of two lines perpendicular to the line of intersection.

§ 11. To find the equations of a space, orthogonal to a given r-space and passing through a given point.

Let P $(a_1, a_2, \dots a_n)$ be the given point and let the equations of the given r-space be

$$l_{j\,1}x_1 + l_{j\,2}x_2 + l_{j\,3}x_3 + \quad \dots \quad + l_{j\,n}x_n + \text{k} = 0 \quad \dots \quad (1)$$

$$(j=1, 2, 3, \dots\dots\dots n-r).$$

Therefore, the equations of the parallel r-space passing through P are $l_{j\,1}(x_1 - a_1) + l_{j\,2}(x_2 - a_2) + \ \dots\ + l_{j\,n}(x_n - a_n) = 0 \quad \dots \quad (2)$

$$(j=1, 2, 3, \quad \dots \quad \dots \quad n-r)$$

Now, every line in a space orthogonal to a given space is perpendicular to every line in the given space. But since there are only $(n-r)$ independent normals which can be drawn to an r-space through a given point in it, any other line drawn through the point and orthogonal to the r-space must have its direction-cosines as linear functions of the direction-cosines of the $(n-r)$ independent lines.

Let Q $(x_1, x_2, x_3 \dots\dots x_n)$ be any point in the orthogonal space. Then the direction-cosines of the line PQ are proportional to $x_1 - a_1$, $x_2 - a_2$, $x_3 - a_3 \dots\dots x_n - a_n$.

Hence, from what has been said above, we must have

$$\rho(x_i - a_i) = \lambda_1 l_{1i} + \lambda_2 l_{2i} + \lambda_3 l_{3i} + \dots\dots + \lambda_{n-r}\, l_{n-r\,i}\,; \dots\dots\dots (3)$$

$$(i = 1, 2, 3, \dots\dots n)$$

Eliminating the $(n-r+1)$ quantities ρ, λ_1, $\lambda_2, \dots\dots \lambda_{n-r}$ from these equations we get

$$\begin{Vmatrix} x_1 - a_1 & x_2 - a_2 & \dots & x_n - a_n \\ l_{11} & l_{12} & \dots & l_{1n} \\ l_{21} & l_{22} & \dots & l_{2n} \\ \dots & \dots & \dots & \dots \\ l_{n-r\,1} & l_{n-r\,2} & \dots & l_{n-r\,n} \end{Vmatrix} = 0$$

This gives r independent linear equations which determine an $(n-r)$-space, passing through the given point, orthogonal to the given r-space.

§12. To find the equations of a perpendicular to an Hyper-space of r-dimensions, drawn from a given point outside it.

Let the given point be P $(a_1, a_2, a_3, \dots\dots a_n)$ and the r-space be given by

$$l_{j1}x_1 + l_{j2}x_2 + \dots\dots + l_{jn}x_n + k = 0 \dots\dots (1)$$

$$(j = 1, 2, 3, \dots\dots n-r).$$

Let Q be the point where the perpendicular through P meets the r-space. Let $(l_1, l_2, \dots\dots l_n)$ be the direction-cosines of PQ. Then we may take its equations to be

$$\frac{x_1 - a_1}{l_1} = \frac{x_2 - a_2}{l_2} = \frac{x_3 - a_3}{l_3} = \dots\dots\dots\dots = \frac{x_n - a_n}{l_n} = r \dots\dots (2)$$

Since PQ is a normal to the r-space at Q, it lies in the normal space at Q and hence its direction-cosines must be representable as linear functions of the direction-cosines of the $(n-r)$ independent normals at Q.

Hence

$$l_i = \lambda_1 l_{1i} + \lambda_2 l_{2i} + \ldots\ldots + \lambda_{n-r} l_{n-r\,i} \quad \ldots\ldots\ldots\ldots(3)$$
$$(i = 1, 2, 3, \ldots\ldots n)$$

Eliminating $(n-r)$ quantities $\lambda_1, \lambda_2, \ldots\ldots \lambda_{n-r}$ between equations (3) we get

$$\left\| \begin{array}{cccc} l_1 & l_2 & l_2 \ldots\ldots\ldots & l_n \\ l_{11} & l_{12} & l_{13} \ldots\ldots\ldots & l_{1n} \\ l_{21} & l_{22} & l_{23} \ldots\ldots\ldots & l_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ l_{n-r\,1} & l_{n-r\,2} & l_{n-r\,3} \ldots\ldots\ldots & l_{n-r\,n} \end{array} \right\| = 0$$

These give r equations involving the n quantities (unknown)

$(l_1, l_2, l_3, \quad \ldots \quad l_n)$ ... ... (4)

Again the coordinates of the point Q may be taken as

$$a_1 + l_1\, r, \quad a_2 + l_2\, r, \quad \ldots \quad a_n + l_n\, r, \text{ (where } r = \text{PQ).}$$

Since Q lies in the r-space we must have, from equations (I) $(n-r)$ equations of the type

$$l_{j1}\,(a_1 + l_1 r) + l_{j2}\,(a_2 + l_2 r) + \ \ldots\ l_{jn}\,(a_n + l_n r) + k = 0.$$

or

$$r\,(l_{j1} l_1 + l_{j2} l_2 + \ldots l_{jn} l_n) + (l_{j1}\, a_1 + l_{j2}\, a_2 + \ldots l_{jn} a_n + k) = 0.$$
$$(j = 1, 2, 3, \ \ldots\ n-r) \quad \ldots \quad \ldots \quad (5)$$

Eliminating r between these $(n-r)$ equations we get a series of $(n-r-1)$ equations involving only the n quantities $l_1, l_2 \ \ldots\ l_n$ ... ... (6)

Hence from equations (4) and (6) we get altogether $(n-1)$ equations to determine the ratios of the n quantities $l_1, l_2, \ldots l_n$.

Thus determining these ratios, we substitute in equations (2) which give the equations of the perpendicular to the r-space.

Cor :—From any of the $(n-r)$ equations (5), we can find the length (r) of this perpendicular and then the coordinates of the point Q, the foot of the perpendicular through P.

§13. To find the coordinates of the foot of the perpendicular, drawn from an outside point, to a 3-space determined by four given points.

Let P $(\alpha_1, \alpha_2, \ldots \alpha_n)$ be the point and let the 3-space be determined by the four given points, whose coordinates are given by (a_r, b_r, c_r, d_r) respectively $(r=1, 2, 3, \ldots n)$.

Let the coordinates of the foot of the perpendicular from P on the 3-space be denoted by $(x_1, x_2, x_3, \ldots x_n)$.

Since this point lies in the 3-space, the coordinates must be expressible in the form

$$x_r=\lambda a_r+\mu b_r+\nu c_r+\rho d_r \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$(r=1, 2, 3, \quad \ldots \quad n)$$

$$1=\lambda+\mu+\nu+\rho \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where λ, μ, ν, ρ are indeterminate.

Now the direction-cosines of the perpendicular from P are proportional to $(x_1-\alpha_1, x_2-\alpha_2, x_3-\alpha_3 \quad \ldots \quad x_n-\alpha_n)$ respectively; and also the perpendicular is at rt. angles to all lines drawn in the 3-space. Now the direction-cosines of lines joining the point a to b, c, d are respectively proportional to a_r-b_r; a_r-c_r; a_r-d_r,

$$(r=1, 2, 3, \quad \ldots \quad n).$$

Hence

$$\left.\begin{aligned}
&\sum_{r=1}^{r=n}(x_r-\alpha_r)\ (a_r-b_r)=0\\
&\sum_{r=1}^{r=n}(x_r-\alpha_r)\ (a_r-c_r)=0\\
&\sum_{r=1}^{r=n}(x_r-\alpha_r)\ (a_r-d_r)=0
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

From (3) we easily deduce

$$\sum_{r=1}^{r=n}(x_r-\alpha_r)a_r=\sum_{r=1}^{r=n}(x_r-\alpha_r)\ b_r=\sum_{r=1}^{r=n}(x_r-\alpha_r)\ c_r$$

$$=\sum_{r=1}^{r=n}(x_r-\alpha_r)\ d_r=\theta \text{ (say).}$$

By substituting the values of x_r from (1) in these equations we obtain

$$\left.\begin{array}{l}
\lambda \sum_1^n a_r^2 + \mu \sum_1^n a_r b_r + \nu \sum_1^n a_r c_r + \rho \sum_1^n a_r d_r - \theta = \sum_1^n a_r \alpha_r \\
\lambda \sum_1^n a_r b_r + \mu \sum_1^n b_r^2 + \nu \sum_1^n b_r c_r + \rho \sum_1^n b_r d_r - \theta = \sum_1^n b_r \alpha_r \\
\lambda \sum_1^n a_r c_r + \mu \sum_1^n b_r c_r + \nu \sum_1^n c_r^2 + \rho \sum_1^n c_r d_r - \theta = \sum_1^n c_r \alpha_r \\
\lambda \sum_1^n a_r d_r + \mu \sum_1^n b_r d_r + \nu \sum_1^n c_r d_r + \rho \sum_1^n d_r^2 - \theta = \sum_1^n d_r \alpha_r \\
\text{also, } \lambda + \mu + \nu + \rho - 0 \cdot \theta = 1
\end{array}\right\} (4)$$

From the above five equations in (4) we can easily find the values of the five unknowns λ, μ, ν, ρ, θ.

Thus,

$$\theta \times \begin{vmatrix}
\sum a_r^2 & \sum a_r b_r & \sum a_r c_r & \sum a_r d_r & 1 \\
\sum a_r b_r & \sum b_r^2 & \sum b_r c_r & \sum b_r d_r & 1 \\
\sum a_r c_r & \sum b_r c_r & \sum c_r^2 & \sum c_r d_r & 1 \\
\sum a_r d_r & \sum b_r d_r & \sum c_r d_r & \sum d_r^2 & 1 \\
1 & 1 & 1 & 1 & 0
\end{vmatrix} =$$

$$\begin{vmatrix}
\sum a_r^2 & \sum a_r b_r & \sum a_r c_r & \sum a_r d_r & \sum a_r \alpha_r \\
\sum a_r b_r & \sum b_r^2 & \sum b_r c_r & \sum b_r d_r & \sum b_r \alpha_r \\
\sum a_r c_r & \sum b_r c_r & \sum c_r^2 & \sum c_r d_r & \sum c_r \alpha_r \\
\sum a_r d_r & \sum b_r d_r & \sum c_r d_r & \sum d_r^2 & \sum d_r \alpha_r \\
1 & 1 & 1 & 1 & 1
\end{vmatrix}$$

where $\sum$ extends always from 1 to n inclusive.

The co-efficient of θ in this is equivalent to the product of the matrices

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n & 0 & 1 \\ b_1 & b_2 & b_3 & \dots & b_n & 0 & 1 \\ c_1 & c_2 & c_3 & \dots & c_n & 0 & 1 \\ d_1 & d_2 & d_3 & \dots & d_n & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n & 1 & 0 \\ b_1 & b_2 & b_3 & \dots & b_n & 1 & 0 \\ c_1 & c_2 & c_3 & \dots & c_n & 1 & 0 \\ d_1 & d_2 & d_3 & \dots & d_n & 1 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & 1 \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}^2 = k^2 \text{ (say)}$$

and the determinant on the right-hand side is equivalent to the product of the matrices

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n & 1 \\ b_1 & b_2 & b_3 & \dots & b_n & 1 \\ c_1 & c_2 & c_3 & \dots & c_n & 1 \\ d_1 & d_2 & d_3 & \dots & d_n & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n & 1 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n & 0 \\ b_1 & b_2 & b_3 & \dots & b_n & 0 \\ c_1 & c_2 & c_3 & \dots & c_n & 0 \\ d_1 & d_2 & d_3 & \dots & d_n & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 1 \end{vmatrix}$$

$= \Sigma \kappa_4 \; P_5$ (say)

$\therefore\ \theta = \Sigma\ \kappa_4\ P_5 / k^2$ ... ... (5)

Similarly the values of λ, μ, ν, ρ can be found.

Now, we have from (1) $x_r = \lambda a_r + \mu b_r + \nu c_r + \rho d_r$.

The denominator in the values of λ, μ, ν, ρ is the same, and, $=k^2$; and the numerators are respectively the co-factors of A, B, C and D in the determinant

$$\begin{vmatrix} 0 & A & B & C & D & 0 \\ \Sigma\, \alpha_r a_r & \Sigma\, a_r^2 & \Sigma\, a_r b_r & \Sigma\, a_r c_r & \Sigma\, a_r d_r & 1 \\ \Sigma\, b_r \alpha_r & \Sigma\, a_r b_r & \Sigma\, b_r^2 & \Sigma\, b_r c_r & \Sigma\, b_r d_r & 1 \\ \Sigma\, c_r \alpha_r & \Sigma\, a_r c_r & \Sigma\, b_r c_r & \Sigma\, c_r^2 & \Sigma\, c_r d_r & 1 \\ \Sigma\, d_r \alpha_r & \Sigma\, a_r d_r & \Sigma\, b_r d_r & \Sigma\, c_r d_r & \Sigma\, d_r^2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

Hence the value of x_r is given by the equation

$$\begin{vmatrix} x_r & a_r & b_r & c_r & d_r & 0 \\ \Sigma\, a_r \alpha_r & \Sigma\, a_r^2 & \Sigma\, a_r b_r & \Sigma\, a_r c_r & \Sigma\, a_r d_r & 1 \\ \Sigma\, b_r \alpha_r & \Sigma\, a_r b_r & \Sigma\, b_r^2 & \Sigma\, b_r c_r & \Sigma\, b_r d_r & 1 \\ \Sigma\, c_r \alpha_r & \Sigma\, a_r c_r & \Sigma\, b_r c_r & \Sigma\, c_r^2 & \Sigma\, c_r d_r & 1 \\ \Sigma\, d_r \alpha_r & \Sigma\, a_r d_r & \Sigma\, b_r d_r & \Sigma\, c_r d_r & \Sigma\, d_r^2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \quad \ldots (6)$$

This gives the co-ordinates of the foot of the perpendicular drawn from the point $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n)$ to the 3-space determined by four given points a, b, c, d.

N.B.—The method adopted is perfectly general and can therefore be used for finding the co-ordinates of the foot of the perpendicular drawn from any external point to a space of any number of dimensions determined by given points.

§ 14. To find the angle between the perpendiculars which can be drawn from two external points P and Q to a 3-space determined by four given points.

Let $P(\alpha_1, \alpha_2, \alpha_2, \ldots \alpha_n)$ and $Q\ (\beta_1, \beta_2, \beta_3, \ldots \beta_n)$ be the two points and let the elements of the 3-space be the same as in the previous article.

Now, the equations (6) of § **13** can be written as

$$k^2 x_r - \Delta_r = 0$$

or,
$$x - \frac{\Delta_r}{k^2} = 0$$

But in order to find the cosine of the angle we require to know the values of $\alpha_r - x_r$ $(r = 1, 2, 3, \ldots n)$, *i.e.* of $\alpha_r - \frac{\Delta_r}{k^2} = \frac{1}{k^2}(k^2\alpha_r - \Delta_r)$. Hence, if in equation (6) of § 13, we put α_r for x, and divide the result by k^2, we obtain

$$\alpha_r - x_r = \frac{1}{k^2}\begin{vmatrix} \alpha_r & a_r & b_r & c_r & d_r & 0 \\ \Sigma a_r\alpha_r & \Sigma a_r^2 & \Sigma a_r b_r & \Sigma a_r c_r & \Sigma a_r d_r & 1 \\ \Sigma b_r\alpha_r & \Sigma a_r b_r & \Sigma b_r^2 & \Sigma b_r c_r & \Sigma b_r d_r & 1 \\ \Sigma c_r\alpha_r & \Sigma a_r c_r & \Sigma b_r c_r & \Sigma c_r^2 & \Sigma c_r d_r & 1 \\ \Sigma d_r\alpha_r & \Sigma a_r d_r & \Sigma b_r d_r & \Sigma c_r d_r & \Sigma d_r^2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{k^2}\phi_r \text{ (say)}$$

Similarly, $\beta_r - \chi_r = \frac{1}{k^2}\phi^1_r$, where χ_r are the co-ordinates of the foot of the perpendicular from Q.

∴ If ψ be the angle between the perpendiculars, we have

$$\cos\psi = \Sigma(\alpha_r - x_r)(\beta_r - \chi_r) / \sqrt{\Sigma(\alpha_r - x_r)^2}\sqrt{\Sigma(\beta_r - \chi_r)^2}$$

$$= \frac{\Sigma\phi_r\,\phi^1_r}{\sqrt{\Sigma\phi^2_r}\,\sqrt{\Sigma\phi^{1\,2}_r}}.$$

§ 15. To find the relation connecting $(n+2)$ points in an n-space.

Multiply the two rectangular arrays of $(n+2)$ columns and $(n+3)$ rows as follows :—

$$\begin{vmatrix} 1 & 0 & & 0 & 0 \\ \Sigma x^2_{11} & -2x_{12} & \ldots & -2x_{1n} & 1 \\ \Sigma x^2_{21} & -2x_{22} & \ldots & -2x_{2n} & 1 \\ \ldots & \ldots & \ldots & \ldots & \\ \Sigma\left(x_{n+2,1}\right)^2 & -2x_{n+2,2} & \ldots & -2x_{n+2,n} & 1 \end{vmatrix} \times$$

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & x_{11} & \ldots & x_{1n} & \Sigma x^2_{11} \\ 1 & x_{21} & \ldots & x_{2n} & \Sigma x^2_{21} \\ \ldots & & \ldots & \ldots & \ldots \\ 1 & x_{n+2,1} & \ldots & x_{n+2,n} & \Sigma\left(x_{1,n+2}\right)^2 \end{vmatrix}$$

The result of this multiplication must vanish identically.*

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* *Vide*—Burnside and Panton's Theory of Equations, Vol. **II**, § **143**.

Therefore we must have

$$\begin{vmatrix} 0 & 1 & 1 & .. & 1 \\ 1 & 0 & (1,2)^2 & \ldots & (1,n+2)^2 \\ 1 & (2,1)^2 & 0 & \ldots & (2,n+2)^2 \\ 1 & (3,1)^2 & (3,2)^2 & \ldots & (3,n+2)^2 \\ \ldots & \ldots & .. & \ldots & \ldots \\ 1 & (n+2,1)^2 & (n+2,2)^2 & \ldots & 0 \end{vmatrix} = 0 \quad \ldots \quad (1)$$

where, the points being marked with the numbers 1, 2, 3, ... $n+2$, $(1, 2)^2$ stands for the square of the distance between the points 1 and 2; and $(1, 2)^2 \equiv (2, 1)^2$.

Note.—The theorem may be stated in a different form:—

To find the condition that an $(n+2)$th given point may lie in an n-space determined by $(n+1)$ given points.

§ 16. Plückerian co-ordinates of a right line:

We have the following equations of a right line:—

$$\frac{\chi_1-a_1}{l_1}=\frac{\chi_2-a_2}{l_2}=\frac{\chi_3-a_3}{l_3}= \quad \ldots \quad =\frac{\chi_n-a_n}{l_n}$$

By combining these equations in pairs, we obtain the following n equations:—

$$\left.\begin{array}{lllll} l_2\,\chi_1 & -\,l_1\,\chi_2 & -\,a_1\,l_2 & +\,a_2\,l_1 & =0 \\ l_3\,\chi_2 & -\,l_2\,\chi_3 & -\,a_2\,l_3 & +\,a_3\,l_2 & =0 \\ l_4\,\chi_3 & -\,l_3\,\chi_4 & -\,a_3\,l_4 & +\,a_4\,l_3 & =0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ l_n\,\chi_{n-1} & -\,l_{n-1}\,\chi_n & -\,a_{n-1}\,l_n & +\,a_n\,l_{n-1} & =0 \\ l_1\,\chi_n & -\,l_n\,\chi_1 & -\,a_n\,l_1 & +\,a_1\,l_n & =0 \end{array}\right\}$$

$$\text{If we put } \left\{ \begin{array}{lll} a_1 l_2 & - a_2 l_1 & = a_{12} \\ a_2 l_3 & - a_3 l_2 & = a_{23} \\ \dots & \dots & \dots \\ a_{n-1} l_n & - a_n l_{n-1} & = a_{n,\, n-1} \\ a_n l_1 & - a_1 l_n & = a_{n,\, 1} \end{array} \right.$$

the above equations may be written as

$$\left. \begin{array}{lll} l_2 \chi_1 & - l_1 \chi_2 & = a_{12} \\ l_3 \chi_2 & - l_2 \chi_3 & = a_{23} \\ \dots & \dots & \dots \\ l_n \chi_{n-1} & - l_{n-1} \chi_n & = a_{n,\, n-1} \\ l_1 \chi_n & - l_n \chi_1 & = a_{n,\, 1} \end{array} \right\} \quad \dots \quad (1)$$

From equations (1) it is easily seen that the following relation must hold :—

$$\frac{a_{12}}{l_1 l_2} + \frac{a_{23}}{l_2 l_3} + \frac{a_{34}}{l_3 l_4} + \dots + \frac{a_{n,\, n-1}}{l_n l_{n-1}} + \frac{a_{n,\, 1}}{l_n l_1} = 0 \quad \dots \quad (2)$$

Now, we have "$2n$ quantities" $(a_{12}, a_{23}, \dots a_{n,\, n-1}, a_{n,\, 1})$ and $(l_1, l_2, l_3, \dots l_n)$ which serve to determine the position of the right line, provided the relation (2) holds.

These $2n$ quantities we shall call the "$2n$ co-ordinates" or the Plückerian co-ordinates of a line.

§ 17. To express the shortest distance between two lines in an n-space in terms of their Plückerian co-ordinates.

Since two points determine a right line, we have altogether four points given to determine the two lines. Through these four points a 3-space can be drawn (§ 8) which will contain both these lines. Again, since the shortest distance meets both the lines, it has two points lying in the 3-space and consequently lies wholly in it. Thus the problem is reduced to one in a 3-space.

In the 3-space, *i.e.*, in the ordinary space of three dimensions, we have—

The equations of the lines being
$$\begin{cases} \dfrac{x_1-a_1}{l_1}=\dfrac{x_2-a_2}{l_2}=\dfrac{x_3-a_3}{l_3}, \\[2ex] \dfrac{x_1-a_1'}{m_1}=\dfrac{x_2-a_2'}{m_2}=\dfrac{x_3-a_3'}{m_3} \end{cases}$$

δ being the shortest distance and θ the angle between the lines

$$*\delta \sin\theta = \begin{vmatrix} a_1-a_1' & a_2-a_2' & a_3-a_3' \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} \quad \dots \quad (1)$$

Now, any point on the first line has for coordinates

$$a_1+l_1r_1, \quad a_2+l_2r_1, \quad a_3+l_3r_1 ;$$

and the coordinates of any point on the second line are

$$a_1'+m_1r_2 , \quad a_2'+m_2r_2 , \quad a_3'+m_3r_2 ;$$

* (Salmon's Solid Geometry § 51).

∴ The volume of the tetrahedron determined by the four points is given by

$$V_3 = \frac{1}{3!}\begin{vmatrix} a_1 + l_1 r_1 & a_2 + l_2 r_1 & a_3 + l_3 r_1 & 1 \\ a_1' + m_1 r_2 & a_2' + m_2 r_2 & a_3' + m_3 r_2 & 1 \\ a_1 & a_2 & a_3 & 1 \\ a_1' & a_2' & a_3' & 1 \end{vmatrix}$$

$$= \frac{r_1 r_2}{6}\begin{vmatrix} a_1 - a_1' & a_2 - a_3' & a_3 - a_3' \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}$$

$$= \frac{r_1 r_2}{6} \,.\, \delta \sin\theta, \text{ by (1).} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad . \quad (2)$$

Again, in an n-space, the content of the "finite join" of the four points is given by

$$*\left(3!V_3\right)^2 = \Sigma \begin{vmatrix} a_1 + l_1 r_1 & a_2 + l_2 r_1 & a_3 + l_3 r_1 \\ a_1' + m_1 r_2 & a_2' + m_2 r_2 & a_3' + m_3 r_2 \\ a_1 & a_2 & a_3 \\ a_1' & a_1' & a_3' \end{vmatrix}^2$$

$$\text{or} \left(3!V_3\right)^2 = r_1^2 r_2^2 \,\Sigma \begin{vmatrix} a_1 - a_1' & a_2 - a_2' & a_3 - a_3' \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}^2$$

$$= r_1^2 r_2^2 \,\Sigma\left(l_1 \alpha_{23}' + l_2 \alpha_{13}' + l_3 \alpha_{12}' + m_1 \alpha_{23} + m_2 \alpha_{13} + m_3 \alpha_{12}\right)^2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where $(l_1, l_2, l_3, \ldots l_n)$ and $(\alpha_{12}, \alpha_{23}, \alpha_{34}, \ldots)$ are the Plückerian coordinates of one line and $(m_1, m_2, \ldots m_n$ and $\alpha_{12}', \alpha_{13}', \ldots)$ are those of the other line.

From (2) and (3) we find

$$\delta \sin\theta = \sqrt{\Sigma(l_1 \alpha_{23}' + l_2 \alpha_{13}' + l_3 \alpha_{12}' + m_1 \alpha_{23} + m_2 \alpha_{13} + m_3 \alpha_{12})^2} \quad (A)$$

This gives the shortest distance between the two lines in terms of their Plückerian coordinates.

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* Vide—Proc. of the London Math. Soc. Vols. XVIII and XIX.

When we put $n=3$, we get the formula in our ordinary Geometry of three dimensions, viz :—

$$* \; \delta \sin \theta = l_1 a_{23}' + l_2 a_{13}' + l_3 a_{12}' + m_1 a_{23} + m_2 a_{13} + m_3 a_{12}$$

Cor. The two lines intersect if $\delta=0$

§ 18. To find the content of the "join" of (n+1) given points in an n-space in terms of the mutual distances between the points.

Let the points be denoted by the numerals 1, 2, 3,......n, $n+1$; and let their coordinates be given by $x_i^{(j)}$ ($i=1, 2, 3, 4,\ldots n$; $j=1, 2, 3,\ldots n, n+1$)

Now, we have

$$\left(n!\ V_n\right) = \begin{vmatrix} 1 & x_1^{(1)} & x_2^{(1)} & \ldots & x_n^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & \ldots & x_n^{(2)} \\ & \ldots & \ldots & & \ldots \\ 1 & x_1^{(n+1)} & x_2^{(n+1)} & \ldots & x_n^{(n+1)} \end{vmatrix}$$

This may be written as

$$(-1)^{n+1}(n!V_n) = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & x_1^{(1)} & x_2^{(1)} & \ldots x_n^{(1)} & \Sigma\left(x^{(1)}\right)^2 \\ 1 & x_1^{(2)} & x_2^{(2)} & \ldots x_n^{(2)} & \Sigma\left(x^{(2)}\right)^2 \\ & \ldots & \ldots & & \ldots \\ 1 & x_1^{(n+1)} & x_2^{(n+1)} & \ldots x_n^{(n+1)} & \Sigma\left(x^{(n+1)}\right)^2 \end{vmatrix} \equiv K_n^{(n+1)} \text{ (say)}$$

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* Cf. Salmon's Solid Geometry, Ex. § 53

But we have

$$2^n\left(n!\,V_n\right)^2 = 2^n\left(K_n^{(n+1)}\right)^2 = 2^n . K_n^{(n+1)} . \; K_n^{(n+1)}$$

$$= \left|\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \Sigma\left(\overset{(\ )}{x}\right)^2 & -2\overset{(1)}{x}_1 & -2\overset{(1)}{x}_2 & \ldots -2\overset{(1)}{x}_n & 1 \\ \Sigma\left(\overset{(2)}{x}\right)^2 & -2\overset{(2)}{x}_1 & -2\overset{(2)}{x}_2 & \ldots -2\overset{(2)}{x}_n & 1 \\ \ldots & \ldots & \ldots & \ldots & \\ \Sigma\left(\overset{(n+1)}{x}\right)^2 & -2\overset{(n+1)}{x}_1 & -2\overset{(n+1)}{x}_2 & \ldots -2\overset{(n+1)}{x}_n & 1 \end{array}\right| \times$$

$$\left|\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 1 & \overset{(1)}{x}_1 & \overset{(1)}{x}_2 \;\cdots & \overset{(1)}{x}_n & \Sigma\left(\overset{(1)}{x}\right)^2 \\ 1 & \overset{(2)}{x}_1 & \overset{(2)}{x}_2 \;\cdots & \overset{(2)}{x}_n & \Sigma\left(\overset{(2)}{x}\right)^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \overset{(n+1)}{x}_1 & \overset{(n+1)}{x}_2 \;\cdots & \overset{(n+1)}{x}_n & \Sigma\left(\overset{(n+1)}{x}\right)^2 \end{array}\right|$$

$$= \left|\begin{array}{ccccc} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & \left(\overset{(1)}{x}\,\overset{(2)}{x}\right)^2 & \ldots & \left(\overset{(1)}{x}\,\overset{(n+1)}{x}\right)^2 \\ 1 & \left(\overset{(1)}{x}\,\overset{(2)}{x}\right)^2 & 0 & \ldots & \left(\overset{(2)}{x}\,\overset{(n+1)}{x}\right)^2 \\ \ldots & \ldots & \ldots & \ldots & \\ 1 & \left(\overset{(1)}{x}\,\overset{(n+1)}{x}\right)^2 & \left(\overset{(2)}{x}\,\overset{(n+1)}{x}\right)^2 & \ldots & 0 \end{array}\right|$$

$$= \left|\begin{array}{ccccc} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & (2,1)^2 & \ldots & (1,n+1)^2 \\ 1 & (1,2)^2 & 0 & \ldots & (2,n+1)^2 \\ & \ldots & \ldots & \ldots & \ldots \\ 1 & (1,n)^2 & (2,n)^2 & \ldots & (n,n+1)^2 \\ 1 & (1,n+1)^2 & (2,n+1)^2 & \ldots & 0 \end{array}\right|$$

where $(r,s)^2$ stands for the square of the distance between the points denoted by r and s; and $(r,s)^2 \equiv (s,r)^2$.

§19. To express the content of the "join" of (n+1) given points in terms of their oblique coordinates.

Take one of the given points as origin, and let the direction-cosines of the n lines joining the origin with the remaining n points, with reference to a system of rectangular axes through the origin, be given by $\overset{(j)}{l}_i \left(i=1,2,3,\ldots n\,;\, j=1,2,3,\ldots n\right)$

Let the coordinates of the n points, with reference to a system of oblique axes through the origin, be denoted by $\overset{(j)}{\xi}_i$ and the rectangular coordinates of the same points be denoted by $\overset{(j)}{x}_i \left(i=1,\ 2,\ 3,\ldots n\,;\ j=1,\ 2,\ 3,\ldots n\right)$

We shall designate the oblique axes by the numerals 1, 2, 3, ...n, and the angle between the rth and the sth axes by $\overset{\wedge}{rs}$, etc.

Let O be the origin and P any one of the remaining n points. Now, the projection of OP on the axis of x_1 is $\overset{(j)}{x}_1$; but it is equal to the sum of the projections on the *same* axis of x_1 of lengths equal to the oblique coordinates of P measured respectively along the oblique axes.

Therefore we must have

$$\overset{(j)}{x}_i = \overset{(1)}{l}_i\,\overset{(j)}{\xi}_1 + \overset{(2)}{l}_i\,\overset{(j)}{\xi}_2 + \;\cdots\; + \overset{(n)}{l}_i\,\overset{(j)}{\xi}_n \qquad \ldots \quad (A)$$

$$(j=1.\ 2,\ 3,\ldots n\,;\ i=1,\ 2,\ 3,\ldots n)$$

Thus we have expressed the rectangular coordinates of the points in terms of their oblique coordinates.

Now, the content V_n is given by

$$n!\,V_n = \begin{vmatrix} \overset{(1)}{x}_1 & \overset{(1)}{x}_2 & \overset{(1)}{x}_3 & \ldots & \overset{(1)}{x}_n \\ \overset{(2)}{x}_1 & \overset{(2)}{x}_2 & \overset{(2)}{x}_3 & \ldots & \overset{(2)}{x}_n \\ \overset{(3)}{x}_1 & \overset{(3)}{x}_2 & \overset{(3)}{x}_3 & \ldots & \overset{(3)}{x}_n \\ \ldots & & \ldots & & \ldots \\ \overset{(n)}{x}_1 & \overset{(n)}{x}_2 & \overset{(n)}{x}_3 & \ldots & \overset{(n)}{x}_n \end{vmatrix} \qquad \ldots \qquad \ldots \qquad (1)$$

If we substitute the values of x's in (1) from the Scheme (A) we obtain

$$n!\,V_n=\begin{vmatrix} \overset{(1)}{l_1}\overset{(1)}{\xi_1}+\overset{(2)}{l_1}\overset{(1)}{\xi_2}+\dots, & \overset{(1)}{l_2}\overset{(1)}{\xi_1}+\overset{(2)}{l_2}\overset{(1)}{\xi_2}+\dots, & \dots\overset{(1)}{l_n}\overset{(1)}{\xi_1}+\overset{(2)}{l_n}\overset{(1)}{\xi_2}+\dots \\ \overset{(1)}{l_1}\overset{(2)}{\xi_1}+\overset{(2)}{l_1}\overset{(2)}{\xi_2}+\dots, & \overset{(1)}{l_2}\overset{(2)}{\xi_1}+\overset{(2)}{l_2}\overset{(2)}{\xi_2}+\dots, & \dots\overset{(1)}{l_n}\overset{(2)}{\xi_1}+\overset{(2)}{l_n}\overset{(2)}{\xi_2}+\dots \\ \overset{(1)}{l_1}\overset{(3)}{\xi_1}\mp\overset{(2)}{l_1}\overset{(3)}{\xi_2}+\dots, & \overset{(1)}{l_2}\overset{(3)}{\xi_1}+\overset{(2)}{l_2}\overset{(3)}{\xi_2}+\dots . & \dots\overset{(1)}{l_n}\overset{(3)}{\xi_1}+\overset{(2)}{l_n}\overset{(3)}{\xi_2}+\dots \\ \dots & \dots\qquad\dots & \dots \\ \overset{(1)}{l_1}\overset{(n)}{\xi_1}+\overset{(2)}{l_1}\overset{(n)}{\xi_2}+\dots, & \overset{(1)}{l_2}\overset{(n)}{\xi_1}+\overset{(2)}{l_2}\overset{(n)}{\xi_2}+-, & \dots\overset{(1)}{l_n}\overset{(n)}{\xi_1}+\overset{(2)}{l_n}\overset{(n)}{\xi_2}+\dots \end{vmatrix}$$

If we square both sides of this, since $\cos\widehat{rs}=\sum\overset{(r)}{l_t}\,\overset{(s)}{l_t}$, we find on simplification—

$$n!\,V_n=\begin{vmatrix} \overset{(1)}{\xi_1} & \overset{(1)}{\xi_2} & \dots & \overset{(1)}{\xi_n} \\ \overset{(2)}{\xi_1} & \overset{(2)}{\xi_2} & \dots & \overset{(2)}{\xi_n} \\ \overset{(3)}{\xi_1} & \overset{(3)}{\xi_2} & \dots & \overset{(3)}{\xi_n} \\ \dots & \dots & & \\ \overset{(n)}{\xi_1} & \overset{(n)}{\xi_2} & \dots & \overset{(n)}{\xi_n} \end{vmatrix}\times\begin{vmatrix} 1 & \cos\widehat{21} & \cos\widehat{31} & \dots & \cos\widehat{n1} \\ \cos\widehat{12} & 1 & \cos\widehat{32} & \dots & \cos\widehat{n2} \\ \cos\widehat{13} & \cos\widehat{23} & 1 & \dots & \cos\widehat{n3} \\ \dots & \dots & \dots & & \dots \\ \cos\widehat{1n} & \cos\widehat{2n} & \cos\widehat{3n} & \dots & 1 \end{vmatrix}$$

Thus we have expressed V_n in terms of the oblique coordinates of the given points.

* § **20**. We have already seen that a system of $(n\text{-}r)$ independent linear equations are required to determine an r-space, or, in the language of Prof. Cayley,—"a system of $(n\text{-}r)$-fold relations determine an r-omal. Any number of one-fold relations, whether independent or dependent, and if more than n of them whether compatible or incompatible, is termed a **Plexus**, viz:—if the number of one-fold relations be θ, then the "**Plexus**" is θ-fold. A θ-fold Plexus constitutes a relation which is at most θ-fold but which may be less than θ-fold."

Thus the system of equations determining an r-space forms an $(n\text{-}r)$-fold Plexus. But the equations may not all be

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*** This article has been taken from Cayley's Paper—Memoir on Abstract Geometry. Vide—Collected Papers.**

independent *i.e.* one or more of them may be derived from others by algebraic operations; in that case the *Plexus* is not $(n\text{-}r)$-fold, but less; and hence the space is of dimensions greater than r.

We have so far spoken of linear relations constituting a *Plexus*. But the relations may be of any order. In every case, a system of θ relations will form a θ-fold *Plexus*.

From what has been just observed, the idea of Involution can easily be extended to *Plexus*. Any one-fold relation implied in a given r-fold Plexus is said to be in "Involution" with the r-fold relation; and so in a system of one-fold relations if any relation be implied in the other relations *i. e.*, in the relation aggregated of the other relations, then the system is said to be in Involution. A system not in involution is said to be "asyzygetic"

Convolution: Any $(r+1)$ or more or all the relations of the asyzygetic system are in convolution, *i. e.*, any relation of the system is alternately implied in the aggregate of the remaining relations or indeed in the aggregate of any r relations (not being themselves in convolution) of the remaining relations of the asyzygetic system. It may be added that besides the relations of the system there is no one-fold relation alternately implied in the "*asyzygetic*" system

N.B.—The introduction of the ideas of Involution and Convolution in higher Geometries is due to Prof. A. Cayley, in his "Memoir on Abstract Geometry." The ideas will be clearly understood from the following illustration:—

Let the functions or equations $P=o$, $Q=o$, $R=o$, etc., form a Plexus. Then, if identically we have $AP+BQ+CR+\ldots=o$, where A,B,C,...are integral functions of coordinates and where some one of these functions is a constant (A say), then the system P,Q,R,... are in **Involution**, or more accurately, $P=o$ is in involution with the remaining functions $Q=o$, $R=o,\ldots$. But when A,B,C,...are no one of them constant, then we have a **Convolution**. If $P=o$ is in involution with $Q=o$, $R=o,\ldots$, then $P=o$ is implied in these equations and the relations ($Q=o$, $R=o,\ldots$) and ($P=o$, $Q=o$, $R=o,\ldots$) are equivalent. But in the case of convolution $AP+BQ+CR+\ldots=o$, relation of equations $Q=o$, $R=o,\ldots$ imply $AP=o$

i. e. $A=o$ or $P=o$, *i. e.* the relation $(Q=o, R=o,\ldots)$ is a relation composed of two relations $(A=o, Q=o, R=o,\ldots)$ and $(P=o, Q=o, R=o,\ldots)$.

A further extension of the Theory of Involution will be found in Cayley's Paper "on the Theory of Involution"—Trans. of the Camb. Phil. Soc. 1866. The idea of Involution can also be extended to Higher Geometries after Fiedler —"Die Darstellende Geometrie," and Clebsch—"Vorlesungen über Geometrie."

CHAPTER II —Inclinations of Spaces.

§ **21**. We have seen § **4** that if θ be the angle between two lines whose direction-cosines are

$$(l_1, l_2 \;\dots\; l_n) \text{ and } (l_1', l_2', l_3' \;\dots\; l_n') \text{ respectively,}$$

then
$$\cos\theta = l_1 l_1' + l_2 l_2' + l_3 l_3' + \;\dots\; + l_n l_n'$$

and
$$1 = l_1^2 + l_2^2 + l_3^2 + \;\dots\; + l_n^2$$

$$1 = l_1'^2 + l_2'^2 + l_3'^2 + \;\dots\; + l_n'^2$$

Therefore,

$$\sin^2\theta = 1 - \cos^2\theta = \sum_{i=1}^{i=n} l_i^2 . \sum l_i'^2 - \left(\sum_{i=1}^{i=n} l_i l_i'\right)^2$$

$$= \begin{vmatrix} l_1 & l_2 \\ l_1' & l_2' \end{vmatrix}^2 + \begin{vmatrix} l_1 & l_3 \\ l_1' & l_3' \end{vmatrix}^2 + \;\dots$$

$$= \sum \begin{vmatrix} l_r & l_s \\ l_r' & l_s' \end{vmatrix}^2 \; (r = 1, 2, 3, \dots n \;;\; s = r+1, r+2, r+3, \dots n)$$

$$= \sum L_r^2. \text{ (say)}$$

§ **22. To prove that the square of the area of a plane triangle in an n-space is equal to the sum of the squares of its projections on the co-ordinate planes determined by each pair of co-ordinate axes.**

Let OBC be the triangle, O being the origin. Let the coordinates of B and C be x_i and y_i $(i = 1, 2, 3, \;\dots\; n)$ respectively.

Let $\overline{OB} = \lambda$ and $\overline{OC} = \mu$ and the direction-cosines of OB and OC be l_i and l_i' respectively.

Let S represent the area OBC and $S_{r,s}$ that of its projection on the coordinate plane determined by the rth and the sth axes.

If θ be the angle between OB and OC, we have

$$(2S)^2 = \lambda^2\mu^2\sin^2\theta$$

$$= \lambda^2\mu^2 \sum L^2_{r,s}\text{ , by § 21.}$$

$$= \sum \begin{vmatrix} \lambda l_r & \lambda l_s \\ \mu l_r' & \mu l_s' \end{vmatrix}^2 = \sum \begin{vmatrix} x_r & x_s \\ y_r & y_s \end{vmatrix}^2$$

$$= \sum \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_r & x_s \\ 1 & y_r & y_s \end{vmatrix}^2, \quad (r=1, 2, 3, \ldots n;\ s=r+1, r+2, r+3, \ldots n)$$

But $\begin{vmatrix} 1 & 0 & 0 \\ 1 & x_r & x_s \\ 1 & y_r & y_s \end{vmatrix}$ = twice the area of the projection of OBC on the plane passing through the rth and sth axes.

and ∴ $= 2S_{r,s}$

$$\therefore\ (2S)^2 = \sum (2S_{r,s})^2$$

$$\therefore\ S^2 = \sum S^2_{r,s}$$

§ 23. Two planes being given, to find the minimum angle or angles between any two lines, one in each of the given planes.

Let us suppose that the planes are defined by two lines in each, all drawn through a common origin. Let the lines in one plane have direction-cosines l_i and l_i' respectively and those in the other plane have m_i and m_i' for direction-cosines. $(i=1, 2, 3, \ \ldots\ n)$.

Take any two lines λ and μ in the two planes and let their direction-cosines be λ_i and μ_i $(i=1,2,3, \ \ldots\ n)$ respectively.

If θ be the angle between λ and μ, we must have

$$\cos\theta = \sum \lambda_i\mu_i \qquad \ldots \qquad .. \qquad (1)$$

Again, since λ lies in the plane of (l, l') we may take

$$\lambda_i = Al_i + Bl_i'$$

$$(i=1, 2, 3, \dots n)$$

$$1 = \Sigma \lambda_i^2 = A^2+B^2+2AB\,(ll') \qquad \dots \qquad (2)$$

where A and B are indeterminate multipliers, and

$$(ll') \equiv \text{cosine of the angle } \widehat{ll'}.$$

Similarly, since μ lies in the plane of (m, m'), we have

$$\mu_i = Cm_i + Dm_i', \quad (i=1, 2, 3, \dots n)$$

$$1 = \Sigma \mu_i^2 = C^2+D^2+2CD\,(mm') \qquad \dots \qquad (3)$$

where C and D are multipliers.

Equation (1) may be written as—

$$\text{Cos}\,\theta = \sum_1^n \lambda_i \, \mu_i$$

$$= \sum_1^n (Al_i+Bl_i')\,(Cm_i+Dm_i')$$

$$= AC(lm)+AD(lm')+BC(l'm)+BD(l'm') \quad \dots \quad (4)$$

Differentiating (2), (3), and (4), we have

$$o = \{A+B\,(ll')\}\delta A+\{B+A(ll')\}\delta B, \qquad \dots \qquad (5)$$

$$o = \{C+D(mm')\}\delta C+\{D+C(mm')\}\delta D \qquad \dots \qquad (6)$$

$$o = \{C(lm)+D(lm')\}\delta A+\{C(l'm)+D(l'm')\}\delta B$$

$$+\{A(lm)+B(l'm)\}\delta C+\{A(lm')+B(l'm')\}\delta D, \quad \dots \quad (7)$$

Now, we may keep one of the lines λ and μ fixed and vary the other. Suppose λ is fixed; A and B are constant, so that

$$\delta A=\delta B=0$$

∴ Comparing equations (6) and (7) we have

$$C+D(mm')=k\{A(lm)+B(l'm)\} \qquad \dots \qquad (8)$$

$$D+C(mm')=k\{A(lm')+B(l'm'\} \qquad \dots \qquad (9)$$

where k is an indeterminate multiplier.

Similarly, keeping μ fixed and varying λ we obtain

$$C(lm)+D(lm')=k'\{A+B(ll')\} \quad \ldots \quad (10)$$

$$C(l'm)+D(l'm')=k'\{B+A(ll')\} \quad \ldots \quad (11)$$

where k' is another multiplier.

Multiplying (8) by C and (9) by D, and adding we get

$$k \cos\theta=1 \text{ or } k=1/\cos\theta.$$

Again, multiplying (10) by A and (11) by B, and adding we obtain $k'=\cos\theta$.

Now, the four equations (8), (9), (10) and (11) may be written as:—

$$\begin{cases} A(lm) + B(l'm) - C\cos\theta - D\cos\theta(mm')=o \\ A(lm') + B(l'm') - C\cos\theta(mm') - D\cos\theta = o \\ A\cos\theta + B(ll')\cos\theta - C(lm) - D(lm') = o \\ A(ll')\cos\theta + B\cos\theta - C(l'm) - D(l'm') = o \end{cases}$$

Eliminating the four quantities A, B, C, D between these equations we obtain the following determinant equation giving $\cos\theta$:—

$$\begin{vmatrix} \cos\theta & (ll')\cos\theta & (lm) & (lm') \\ \cos\theta(ll') & \cos\theta & (l'm) & (l'm') \\ (lm) & (l'm) & \cos\theta & (mm')\cos\theta \\ (lm') & (l'm') & (mm')\cos\theta & \cos\theta \end{vmatrix} = o \qquad (\alpha)$$

or $\cos^4\theta[ll']^2[mm']^2+H\cos^2\theta+[ll'/mm']^2=o.$

where H stands for the co-efficient of $\cos^2\theta$ in (α).

$[ll'] \equiv$ Sine of the angle $\widehat{ll'}$.

and $[ll'/mm'] \equiv \Sigma \begin{vmatrix} l_r & l_s \\ l'_r & l'_s \end{vmatrix} \begin{vmatrix} m_r & m_s \\ m'_r & m'_s \end{vmatrix}, (r=1,2,3,\ldots n;\ s=r+1, r+2 \ldots n.)$

$$\text{If } [ll'mm']^2=\Sigma \begin{vmatrix} l_1 & l_2 & l_3 & l_1 \\ l_1' & l_2' & l_3' & l_4' \\ m_1 & m_2 & m_3 & m_4 \\ m_1' & m_2' & m_3' & m_4' \end{vmatrix}^2 = \begin{vmatrix} 1 & (ll') & (lm) & (lm') \\ (ll') & 1 & (l'm) & (l'm') \\ (lm) & (ml') & 1 & (mm') \\ (lm') & (l'm') & (mm') & 1 \end{vmatrix}$$

the co-efficient of $\cos^2\theta$ may be written as

$$H \equiv [ll'mm']^2 - [ll']^2[mm']^2 - [ll'/mm']^2$$

and the determinant (α) reduces to

$$\cos^4\theta[ll']^2[mm']^2 - \{[ll']^2[mm']^2 + [ll'/mm']^2 - [ll'mm']^2\}\cos^2\theta + [ll'/mm']^2 = o \quad \ldots \quad \ldots \quad \ldots \quad (\beta)$$

This is a quadratic in $\cos^2\theta$ and therefore θ has two values—θ_1. θ_2 (say). Thus we have

$$\cos^2\theta_1.\cos^2\theta_2 = [ll'/mm']^2/[ll']^2[mm']^2 \quad \ldots \quad (A)$$

and $\cos^2\theta_1 + \cos^2\theta_2 = -H = [ll']^2[mm']^2 + [ll'/mm']^2 - [ll'mm']^2$

If we change all the cosines into sines in (α) by substituting $\cos^2\theta = 1 - \sin^2\theta$, we obtain

$$\sin^2\theta_1.\sin^2\theta_2 = [ll'mm']^2/[ll']^2[mm']^2 \quad \ldots \quad (B)$$

Note: The idea of minimum angles in Higher-space has been introduced by Veronese, as will appear from the following definitions :—

Definition—" Unter dem Winkel zweier beliebiger Halbräume versteht man denjenigen, welcher durch das normale Segment der beiden Halbebenen im Unendlichgrossen der beiden Halbräume gemessen wird."—§. III, Satz IV, Zusatz II.

And thus Veronese proceeds to define angles between two half-spaces—§. 139, Satz IX ;—

" Der Winkel zweier Halbräume ist der kleinste oder der grösste von den Winkeln, welche ein Strahl des einen mit einem Strahl des andern Halbräums und umgekehrt der letztere mit dem ersten macht, je nachdem er kleiner oder grösser als ein Recter ist."

§. 24, The planes being defined by two sets of three given points, to find the minimum angles between them.

Let $\overset{(j)}{x}_i$ ($j=1, 2, 3$; $i=1, 2, 3, \ldots n$) be the points defining one plane and $\overset{(j)}{y}_i$ ($j=1, 2, 3$; $i=1, 2, 3, \ldots n$) be those defining the other.

Also, let the direction-cosines of the lines joining $\overset{(3)}{x}_i$ to $\overset{(1)}{x}_i$ and $\overset{(2)}{x}_i$ be respectively l_i and m_i ($i=1, 2, 3, \ldots n$) and those of the lines joining $\overset{(3)}{y}_i$ to $\overset{(1)}{y}_i$ and $\overset{(2)}{y}_i$ be respectively p_i and q_i ($i=1, 2, 3, \ldots n$).

Then, by §. 23 we have

$$\cos^2\theta_1.\cos^2\theta_2=[lm/pq]^2/[lm]^2[pq]^2 \qquad \ldots \qquad \ldots \ (1)$$

$$=\left[\begin{array}{cccc|cccc} l_1 & l_2 & l_3 & \ldots l_n & p_1 & p_2 & p_3 & \ldots p_n \\ m_1 & m_2 & m_3 & \ldots m_n & q_1 & q_2 & q_3 & \ldots q_n \end{array}\right]^2 \Big/ [lm]^2[pq]^2$$

But if r_1 and r_2 be the lengths $\overset{(3)}{x}\ \overset{(1)}{x}$ and $\overset{(3)}{x}\ \overset{(2)}{x}$ and r_1', r_2' be those of the lines $\overset{(3)}{y}\ \overset{(1)}{y}$, $\overset{(3)}{y}\ \overset{(2)}{y}$ respectively, we have

$$\left.\begin{array}{l} \dfrac{\overset{(3)}{x}_1-\overset{(1)}{x}_1}{l_1}=\dfrac{\overset{(3)}{x}_2-\overset{(1)}{x}_2}{l_2}=\ldots=\dfrac{\overset{(3)}{x}_n-\overset{(1)}{x}_n}{l_n}=r_1 \\ \dfrac{\overset{(3)}{x}_1-\overset{(2)}{x}_1}{m_1}=\dfrac{\overset{(3)}{x}_2-\overset{(2)}{x}_2}{m_2}=\ldots=\dfrac{\overset{(3)}{x}_n-\overset{(2)}{x}_n}{m_n}=r_2 \end{array}\right\} \ldots (2)$$

and

$$\left.\begin{array}{l} \dfrac{\overset{(3)}{y}_1-\overset{(1)}{y}_1}{p_1}=\dfrac{\overset{(3)}{y}_2-\overset{(1)}{y}_2}{p_2}=\ldots=\dfrac{\overset{(3)}{y}_n-\overset{(1)}{y}_n}{p_n}=r_1' \\ \dfrac{\overset{(3)}{y}_1-\overset{(2)}{y}_1}{q_1}=\dfrac{\overset{(3)}{y}_2-\overset{(2)}{y}_2}{q_2}=\ldots=\dfrac{\overset{(3)}{y}_n-\overset{(2)}{y}_n}{q_n}=r_2' \end{array}\right\} \ldots (3)$$

Substituting the values of l's, m's, &c. obtained from (2) and (3) in the formula (1) for $\cos^2\theta_1.\cos^2\theta_2$, we obtain

$$\cos^2\theta_1.\cos^2\theta_2=\frac{\left[\begin{array}{ccc|ccc} \overset{(3)}{x}_1-\overset{(1)}{x}_1 & \overset{(3)}{x}_2-\overset{(1)}{x}_2 & \ldots & \overset{(3)}{y}_1-\overset{(1)}{y}_1 & \overset{(3)}{y}_2-\overset{(1)}{y}_2 & \ldots \\ \overset{(3)}{x}_1-\overset{(2)}{x}_1 & \overset{(3)}{x}_2-\overset{(2)}{x}_2 & \ldots & \overset{(3)}{y}_1-\overset{(2)}{y}_1 & \overset{(3)}{y}_2-\overset{(2)}{y}_2 & \ldots \end{array}\right]^2}{\Sigma\begin{vmatrix} \overset{(3)}{x}_1-\overset{(1)}{x}_1 & \overset{(3)}{x}_2-\overset{(1)}{x}_2 \\ \overset{(3)}{x}_1-\overset{(2)}{x}_1 & \overset{(3)}{x}_2-\overset{(2)}{x}_2 \end{vmatrix}^2 \Sigma\begin{vmatrix} \overset{(3)}{y}_1-\overset{(1)}{y}_1 & \overset{(3)}{y}_2-\overset{(1)}{y}_2 \\ \overset{(3)}{y}_1-\overset{(2)}{y}_1 & \overset{(3)}{y}_2-\overset{(2)}{y}_2 \end{vmatrix}^2}$$

But $\begin{vmatrix} \overset{(3)}{x}_1-\overset{(1)}{x}_1 & \overset{(3)}{x}_2-\overset{(1)}{x}_2 \\ \overset{(3)}{x}_1-\overset{(2)}{x}_1 & \overset{(3)}{x}_2-\overset{(2)}{x}_2 \end{vmatrix}$ may be written as $\begin{vmatrix} 1 & 0 & 0 \\ 1 & \overset{(3)}{x}_1-\overset{(1)}{x}_1 & \overset{(3)}{x}_2-\overset{(1)}{x}_2 \\ 1 & \overset{(3)}{x}_1-\overset{(2)}{x}_1 & \overset{(3)}{x}_2-\overset{(2)}{x}_2 \end{vmatrix}$

or as $\begin{vmatrix} 1 & 1 & 1 \\ \overset{(1)}{x}_1 & \overset{(2)}{x}_1 & \overset{(3)}{x}_1 \\ \overset{(1)}{x}_2 & \overset{(2)}{x}_2 & \overset{(3)}{x}_2 \end{vmatrix} \equiv (1\ x_1\ x_2)$ say.

Similarly
$$\begin{vmatrix} \overset{(3)}{y}_1-\overset{(1)}{y}_1 & \overset{(3)}{y}_2-\overset{(1)}{y}_2 \\ \overset{(3)}{y}_1-\overset{(2)}{y}_1 & \overset{(3)}{y}_2-\overset{(2)}{y}_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \overset{(1)}{y}_1 & \overset{(2)}{y}_1 & \overset{(3)}{y}_1 \\ \overset{(1)}{y}_2 & \overset{(2)}{y}_2 & \overset{(3)}{y}_2 \end{vmatrix} \equiv (1\;y_1\;y_2) \text{ say.}$$

$$\therefore\ \text{Cos}^2\theta_1 \,.\, \cos^2\theta_2 = \frac{[\Sigma\,(1\;x_1\;x_2)/(1\;y_1\;y_2)\,]^2}{\Sigma\,(1\;x_1\;x_2)^2\;\Sigma(1\;y_1\;y_2)^2} \qquad \ldots (4)$$

Again,

$$Sin^2\theta_1 \,.\, Sin^2\theta_2 = [lmpq]^2 \,/\, [lm]^2\,[pq]^2$$

The numerator becomes

$$\Sigma \begin{vmatrix} \overset{(3)}{x}_1-\overset{(1)}{x}_1 & \overset{(3)}{x}_2-\overset{(1)}{x}_2 & \overset{(3)}{x}_3-\overset{(1)}{x}_3 & \overset{(3)}{x}_4-\overset{(1)}{x}_4 \\ \overset{(3)}{x}_1-\overset{(2)}{x}_1 & \overset{(3)}{x}_2-\overset{(2)}{x}_2 & \overset{(3)}{x}_3-\overset{(2)}{x}_3 & \overset{(3)}{x}_4-\overset{(2)}{x}_4 \\ \overset{(3)}{y}_1-\overset{(1)}{y}_1 & \overset{(3)}{y}_2-\overset{(1)}{y}_2 & \overset{(3)}{y}_3-\overset{(1)}{y}_3 & \overset{(3)}{y}_4-\overset{(1)}{y}_4 \\ \overset{(3)}{y}_1-\overset{(2)}{y}_1 & \overset{(3)}{y}_2-\overset{(2)}{y}_2 & \overset{(3)}{y}_3-\overset{(2)}{y}_3 & \overset{(3)}{y}_4-\overset{(2)}{y}_4 \end{vmatrix}^2$$

which may be written as

$$\Sigma \begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & \overset{(3)}{x}_1-\overset{(1)}{x}_1 & \overset{(3)}{x}_1-\overset{(2)}{x}_1 & \overset{(3)}{y}_1-\overset{(1)}{y}_1 & \overset{(3)}{y}_1-\overset{(2)}{y}_1 \\ 0 & \overset{(3)}{x}_2-\overset{(1)}{x}_2 & \overset{(3)}{x}_2-\overset{(2)}{x}_2 & \overset{(3)}{y}_2-\overset{(1)}{y}_2 & \overset{(3)}{y}_2-\overset{(2)}{y}_2 \\ 0 & \overset{(3)}{x}_3-\overset{(1)}{x}_3 & \overset{(3)}{x}_3-\overset{(2)}{x}_3 & \overset{(3)}{y}_3-\overset{(1)}{y}_3 & \overset{(3)}{y}_3-\overset{(2)}{y}_3 \\ 0 & \overset{(3)}{x}_4-\overset{(1)}{x}_4 & \overset{(3)}{x}_4-\overset{(2)}{x}_4 & \overset{(3)}{y}_4-\overset{(1)}{y}_4 & \overset{(3)}{y}_4-\overset{(2)}{y}_4 \end{vmatrix}^2$$

which again reduces to
$$\Sigma \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \overset{(1)}{x}_1 & \overset{(2)}{x}_1 & \overset{(3)}{x}_1 & \overset{(1)}{y}_1 & \overset{(2)}{y}_1 & \overset{(3)}{y}_1 \\ \overset{(1)}{x}_2 & \overset{(2)}{x}_2 & \overset{(3)}{x}_2 & \overset{(1)}{y}_2 & \overset{(2)}{y}_2 & \overset{(3)}{y}_2 \\ \overset{(1)}{x}_3 & \overset{(2)}{x}_3 & \overset{(3)}{x}_3 & \overset{(1)}{y}_3 & \overset{(2)}{y}_3 & \overset{(3)}{y}_3 \\ \overset{(1)}{x}_4 & \overset{(2)}{x}_4 & \overset{(3)}{x}_4 & \overset{(1)}{y}_4 & \overset{(2)}{y}_4 & \overset{(3)}{y}_4 \end{vmatrix}^2$$

$$\equiv \left[\frac{\overset{(1)}{x}\;\overset{(2)}{x}\;\overset{(3)}{x}\;\overset{(1)}{y}\;\overset{(2)}{y}\;\overset{(3)}{y}}{0,\,1} \right]^2$$

$$\therefore\ \mathrm{Sin}^2\theta_1 . \sin^2\theta_2 = \left[\frac{\overset{(1)}{x}\ \overset{(2)}{x}\ \overset{(3)}{x}\ \overset{(1)}{y}\ \overset{(2)}{y}\ \overset{(3)}{y}}{0,\ 1}\right]^2$$

$$\div \sum(1 x_1 x_2)^2 \sum(1 y_1 y_2)^2 \quad \ldots \quad (5)$$

Note.—We may call $\cos\theta_1 . \cos\theta_2 = \cos\Omega_1$ (say) the "*index of projectivity*" between the two planes and denote it by e_1. Thus $e_1 = \cos\theta_1 . \cos\theta_2$.

Similarly we may write $\sin\theta_1 . \sin\theta_2 = \sin\Omega_2$ and call $\sin\Omega_2$ the "*ratio of conjectivity*," which may be denoted by e_2.

Thus $e_2 = \sin\theta_1 . \sin\theta_2$.

Cor 1. The two given planes will be mutually perpendicular, if either of $\cos\theta_1$ and $\cos\theta_2$ vanishes. They will be "perpendicular of the first kind" or "*simply* perpendicular" if only one of $\cos\theta_1$ and $\cos\theta_2$ vanishes, and the other *does not*; and "perpendicular of the second kind" or "*absolutely* perpendicular" if both vanish. In the latter case *all* lines in one plane are perpendicular to *all* lines in the other. Thus in Hyper-space we meet with a new and more complete kind of perpendicularity, which in our ordinary space would be impossible.

Cor 2. If one of $\sin\theta_1$ and $\sin\theta_2$ vanishes, and the other does not, the two planes intersect in a line. If both vanish, the two planes become coincident.

Cor 3. The two planes will be *isocline i.e.* equally inclined to each other if $\theta_1 = \theta_2$. In this case the two planes have an infinite number of common perpendicular planes on which they cut out equal angles.

Then, $e_1 = \cos^2\theta$ and $e_2 = \sin^2\theta$, where $\theta_1 = \theta_2 = \theta$.

$\therefore\ e_1 + e_2 = \cos^2\theta + \sin^2\theta = 1$.

Thus the condition of isoclinism of two planes is

$$e_1 + e_2 = 1.$$

§ 25. To find the angle between a line and an r-space.

Let the r-space be defined by r lines drawn through the origin, whose direction-cosines are $\overset{(j)}{l}_i$ ($j = 1, 2, 3, \ldots r$; $i = 1, 2, 3, \ldots n$), and let the direction-cosines of the given line be $\overset{(r+1)}{l}_i$ ($i = 1, 2, 3, \ldots n$).

Let $\overset{(j)}{x}_i$ ($j = 1, 2, 3, \ldots r$; $i = 1, 2, 3, \ldots n$), be r points respectively on the r lines.

Then, if V_r be the content of the "join" of the r points and the origin we have :—

$$(r!V_r)^2 = \Sigma \begin{vmatrix} \overset{(1)}{x}_1 & \overset{(1)}{x}_2 & \overset{(1)}{x}_3 & \ldots & \overset{(1)}{x}_r \\ \overset{(2)}{x}_1 & \overset{(2)}{x}_2 & \overset{(2)}{x}_3 & \ldots & \overset{(2)}{x}_r \\ \overset{(3)}{x}_1 & \overset{(3)}{x}_2 & \overset{(3)}{x}_3 & \ldots & \overset{(3)}{x}_r \\ \ldots & & \ldots & & \ldots \\ \overset{(r)}{x}_1 & \overset{(r)}{x}_2 & \overset{(r)}{x}_3 & \ldots & \overset{(r)}{x}_r \end{vmatrix}^2 \equiv H_r^2 \text{ (say)}$$

If h_r be the perpendicular from any point $\overset{(r+1)}{x}_i$ ($i = 1, 2, 3, \ldots n$) on the given line (drawn from the origin) on to the r-space and λ_{r+1} be the radius vector to this point, then we have

$$h_r = H_{r+1}/H_r{}^*$$

If θ be the angle which the line makes with the r-space, we have $\qquad h_r = \lambda_{r+1} \sin\theta$

$$\therefore \lambda_{r+1} \sin\theta = H_{r+1}/H_r.$$

If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_r$ be the radii vectores to the r-points on the lines, then we have $\lambda_j \overset{(j)}{l}_i = \overset{(j)}{x}_i$ ($j = 1, 2, 3, \ldots r$; $i = 1, 2, 3, \ldots n$)

$$\therefore \lambda^2_{r+1} \sin^2\theta = H^2_{r+1}/H^2_r.$$

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* *Vide*—Bulletine of the Calcutta Mathematical Society Vol. I, No. 3, 1909.

$$\text{or } \lambda_{r+1}{}^2\sin^2\theta = \frac{\Sigma\begin{vmatrix} \lambda_1 l_1^{(1)} & \lambda_1 l_2^{(1)} & \dots & \lambda_1 l_r^{(1)} & \lambda_1 l_{r+1}^{(1)} \\ \lambda_2 l_1^{(2)} & \lambda_2 l_2^{(2)} & \dots & \lambda_2 l_r^{(2)} & \lambda_2 l_{r+1}^{(2)} \\ \dots & & \dots & & \dots \\ \lambda_r l_1^{(r)} & \lambda_r l_2^{(r)} & \dots & \lambda_r l_r^{(r)} & \lambda_r l_{r+1}^{(r)} \\ \lambda_{r+1} l_1^{(r+1)} & \lambda_{r+1} l_2^{(r+1)} & \dots & \lambda_{r+1} l_r^{(r+1)} & \lambda_{r+1} l_{r+1}^{(r+1)} \end{vmatrix}^2}{\Sigma\begin{vmatrix} \lambda_1 l_1^{(1)} & \lambda_1 l_2^{(1)} & \dots & \lambda_1 l_r^{(1)} \\ \lambda_2 l_1^{(2)} & \lambda_2 l_2^{(2)} & \dots & \lambda_2 l_r^{(2)} \\ \dots & \dots & & \dots \\ \lambda_r l_1^{(r)} & \lambda_r l_2^{(r)} & \dots & \lambda_r l_r^{(r)} \end{vmatrix}^2}$$

$$\therefore \text{Sin}^2\theta = \frac{\Sigma\begin{vmatrix} l_1^{(1)} & l_2^{(1)} & \dots & l_r^{(1)} & l_{r+1}^{(1)} \\ l_1^{(2)} & l_2^{(2)} & \dots & l_r^{(2)} & l_{r+1}^{(2)} \\ \dots & & \dots & & \dots \\ l_1^{(r)} & l_2^{(r)} & \dots & l_r^{(r)} & l_{r+1}^{(r)} \\ l_1^{(r+1)} & l_2^{(r+1)} & \dots & l_r^{(r+1)} & l_{r+1}^{(r+1)} \end{vmatrix}^2}{\Sigma\begin{vmatrix} l_1^{(1)} & l_2^{(1)} & \dots & l_r^{(1)} \\ l_1^{(2)} & l_2^{(2)} & \dots & l_r^{(2)} \\ \dots & \dots & \dots & \dots \\ l_1^{(r)} & l_2^{(r)} & \dots & l_r^{(r)} \end{vmatrix}^2} \equiv \text{L}_{r+1}{}^2/\text{L}_r{}^2 \text{ (say)}$$

$\therefore$ Sin $\theta = L_{r+1}/L_r$; or in our notation the result may be written as—$\text{Sin}^2\theta = [l^{(1)}\, l^{(2)} \ldots l^{(r)}\, l^{(r+1)}]^2 / [l^{(1)}\, l^{(2)} \ldots l^{(r)}]^2$* ... (1)

Note: If we make use of the method of § 23, we obtain the result in the form of a determinant equation :—

$$\begin{vmatrix} 1 & (l^{(1)}\, l^{(2)}) & (l^{(1)}\, l^{(3)}) & \ldots & (l^{(1)}\, l^{(r+1)}) \\ (l^{(1)}\, l^{(2)}) & 1 & (l^{(2)}\, l^{(3)}) & \ldots & (l^{(2)}\, l^{(r+1)}) \\ (l^{(1)}\, l^{(3)}) & (l^{(2)}\, l^{(3)}) & 1 & \ldots & (l^{(3)}\, l^{(r+1)}) \\ & \ldots & \ldots & \ldots & \\ (l^{(1)}\, l^{(r+1)}) & (l^{(2)}\, l^{(r+1)}) & (l^{(3)}\, l^{(r+1)}) & \ldots & \cos^2\theta \end{vmatrix} = 0 \quad \ldots \quad (II)$$

§ 26. To find the minimum angles between an r-space and a s-space ($n > r > s$).

Let the spaces be defined by lines drawn through the origin. Let $l^{(j)}_i$ $(j=1, 2, 3, \ldots r;\ i=1, 2, 3, \ldots n)$ be the lines of the r-space and $m^{(p)}_i$ $(p=1, 2, 3. \ldots s;\ i=1, 2. 3. \ldots n)$ be those of the s-space.

Any line through the origin lying in the r-space is given by

$$L_i = \lambda_1 l^{(1)}_i + \lambda_2 l^{(2)}_i + \ldots + \lambda_r l^{(r)}_i$$

$$\text{and, } 1 = \Sigma L^2_i = \Sigma \lambda^2_i + 2\, \Sigma\, \lambda_a \lambda_b \left(l^{(a)}\, l^{(b)}\right) \quad \ldots \quad (1)$$

$$(a=1, 2, \ldots r;\ b=a+1, a+2, \ldots r)$$

and any line in the s-space is given by

$$M_i = \mu_1 m^{(1)}_i + \mu_2 m^{(2)}_i + \ldots + \mu_s m^{(s)}_i$$

$$\text{and } 1 = \Sigma\, m^2_i = \Sigma\, \mu^2_i + 2\, \Sigma\, \mu_c \mu_d \left(m^{(c)}\, m^{(d)}\right) \quad \ldots \quad (2)$$

$$(c=1, 2, 3 \ldots s;\ d=c+1, c+2, \ldots s)$$

If θ be the angle between these two lines L and M, we have

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* Campare Veronese—§ 137, Satz. I—Der Winkel, welchen ein Strahl mit einem Raum macht, ist dem Winkel gleich, welchen der Strahl mit seiner Projection auf den Raum macht.

Eng. Trans—The angle which a line makes with a space is equal to the angle which the line makes with its projection on the space.

$$\cos\theta = \Sigma \left(\lambda_1 \overset{(1)}{l_{,}} + \lambda_2 \overset{(2)}{l_{,}} + \dots + \lambda_r \overset{(r)}{l_{,}}\right) . \left(\mu_1 \overset{(1)}{m_{,}} + \mu_2 \overset{(2)}{m_{,}} + \dots + \mu_s \overset{(s)}{m_{,}}\right)$$

$$= \sum_{t=1}^{t=r} \lambda_t \sum_{k=1}^{k=s} \mu_k \left(\overset{(t)}{l} \ \overset{(k)}{m} \right) \qquad \dots \qquad \dots \quad (3)$$

Differentiating equations (1), (2) and (3) we obtain

$$0 = \left\{ \sum_{t=1}^{t=r} \lambda_t \left(\overset{(1)}{l} \ \overset{(t)}{l} \right) \right\} \delta\lambda_1 + \left\{ \sum_{t=1}^{t=r} \lambda_t \left(\overset{(2)}{l} \ \overset{(t)}{l} \right) \right\} \delta\lambda_2 + \dots$$

$$+ \left\{ \sum_{t=1}^{t=r} \lambda_t \left(\overset{(r)}{l} \ \overset{(t)}{l} \right) \right\} \delta\lambda_r .$$

$$= \sum_{k=1}^{k=r} \delta\lambda_k \sum_{t=1}^{t=r} \lambda_t \left(\overset{(k)}{l} \ \overset{(t)}{l} \right) \qquad \dots \qquad \dots \quad (1')$$

$$0 = \sum_{p=1}^{p=s} \delta\mu_p \sum_{q=1}^{q=s} \mu_q \left(\overset{(p)}{m} \ \overset{(q)}{m} \right) \qquad \dots \qquad \dots \quad (2')$$

$$0 = \sum_{t=1}^{t=r} \delta\lambda_t \sum_{k=1}^{k=s} \mu_k \left(\overset{(t)}{l} \ \overset{(k)}{m} \right)$$

$$+ \sum_{t=1}^{t=r} \lambda_t \sum_{k=1}^{k=s} \delta\mu_k \left(\overset{(t)}{l} \ \overset{(k)}{m} \right) \qquad \dots \qquad \dots \quad (3')$$

Then keeping μ fixed and varying λ we have

$$\delta\mu_1 = \delta\mu_2 = \dots = \delta\mu_s = 0$$

$$\therefore \sum_{t=1}^{t=r} \lambda_t \left(\overset{(j)}{l} \ \overset{(t)}{l} \right) = \kappa \sum_{k=1}^{k=s} \mu_k \left(\overset{(j)}{l} \ \overset{(k)}{m} \right)$$

$$(j = 1, 2, 3, \dots r) \qquad \dots \qquad \dots \quad (A)$$

where κ is a multiplier.

Putting $\delta\lambda_1 = \delta\lambda_2 = \dots = \delta\lambda_r = 0$ we obtain

$$\sum_{t=1}^{t=r} \lambda_t \left(\overset{(t)}{l} \ \overset{(k)}{m} \right) = \kappa' \sum_{q=1}^{q=s} \mu_q \left(\overset{(k)}{m} \ \overset{(q)}{m} \right)$$

$$(k = 1, 2, 3, \dots s); \qquad \dots \qquad \dots \quad (B)$$

where κ' is another multiplier.

Multiplying equations (A) respectively by $\lambda_1, \lambda_2, \dots \lambda_r$ and adding we get $\kappa \cos\theta = 1$ and $\therefore \kappa = 1/\cos\theta$.

Similarly from (B) we obtain $\kappa' = \cos\theta$.

Thus replacing κ and κ' in (A) and (B) respectively by $1/\cos\theta$ and $\cos\theta$, we obtain r equations in (A) and s equations in (B), involving the $(r+s)$ unknown quantities $\lambda_1 \; \lambda_2, \dots \lambda_r, \mu_1, \mu_2 \dots \mu_s$.

Hence by eliminating these $(r+s)$ quantities we obtain the following determinant equation giving the values of θ:—

$$\text{I}:\!-\begin{vmatrix}
\cos^2\theta & \cos^2\theta\,(l^{(1)}\,l^{(2)}) & \cdots & \cos^2\theta\,(l^{(1)}\,l^{(r)}) & (l^{(1)}\,m^{(1)}) & (l^{(1)}\,m^{(2)}) & \cdots & (l^{(1)}\,m^{(s)}) \\
\cos^2\theta\,(l^{(1)}\,l^{(2)}) & \cos^2\theta & \cdots & \cos^2\theta\,(l^{(2)}\,l^{(r)}) & (l^{(2)}\,m^{(1)}) & (l^{(2)}\,m^{(2)}) & \cdots & (l^{(2)}\,m^{(s)}) \\
\cos^2\theta\,(l^{(1)}\,l^{(3)}) & \cos^2\theta\,(l^{(2)}\,l^{(3)}) & \cdots & \cos^2\theta\,(l^{(3)}\,l^{(r)}) & (l^{(3)}\,m^{(1)}) & (l^{(3)}\,m^{(2)}) & \cdots & (l^{(3)}\,m^{(s)}) \\
\cos^2\theta\,(l^{(1)}\,l^{(4)}) & \cos^2\theta\,(l^{(2)}\,l^{(4)}) & \cdots & \cos^2\theta\,(l^{(4)}\,l^{(r)}) & (l^{(4)}\,m^{(1)}) & (l^{(4)}\,m^{(2)}) & \cdots & (l^{(4)}\,m^{(s)}) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cos^2\theta\,(l^{(1)}\,l^{(r)}) & \cos^2\theta\,(l^{(2)}\,l^{(r)}) & \cdots & \cos^2\theta & (l^{(r)}\,m^{(1)}) & (l^{(r)}\,m^{(2)}) & \cdots & (l^{(r)}\,m^{(s)}) \\
(l^{(1)}\,m^{(1)}) & (l^{(2)}\,m^{(1)}) & \cdots & (l^{(r)}\,m^{(1)}) & 1 & (m^{(1)}\,m^{(2)}) & \cdots & (m^{(1)}\,m^{(s)}) \\
(l^{(1)}\,m^{(2)}) & (l^{(2)}\,m^{(2)}) & \cdots & (l^{(r)}\,m^{(2)}) & (m^{(1)}\,m^{(2)}) & 1 & \cdots & (m^{(2)}\,m^{(s)}) \\
(l^{(1)}\,m^{(3)}) & (l^{(2)}\,m^{(3)}) & \cdots & (l^{(r)}\,m^{(3)}) & (m^{(1)}\,m^{(3)}) & (m^{(2)}\,m^{(3)}) & \cdots & (m^{(3)}\,m^{(s)}) \\
(l^{(1)}\,m^{(4)}) & (l^{(2)}\,m^{(4)}) & \cdots & (l^{(r)}\,m^{(4)}) & (m^{(1)}\,m^{(4)}) & (m^{(2)}\,m^{(4)}) & \cdots & (m^{(4)}\,m^{(s)}) \\
\cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots \\
\cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots \\
(l^{(1)}\,m^{(s)}) & (l^{(2)}\,m^{(s)}) & \cdots & (l^{(r)}\,m^{(s)}) & (m^{(1)}\,m^{(s)}) & (m^{(2)}\,m^{(s)}) & \cdots & 1
\end{vmatrix}=0$$

or this may again be written as:—

$$
\text{II}:-\begin{vmatrix}
\cos^2\theta & \cos^2\theta\,(m^{(1)}m^{(2)}) & \cdots & \cos^2\theta\,(m^{(1)}m^{(s)}) & (l^{(1)}m^{(1)}) & \cdots & (l^{(r)}m^{(1)}) \\
\cos^2\theta\,(m^{(1)}m^{(2)}) & \cos^2\theta & \cdots & \cos^2\theta\,(m^{(2)}m^{(s)}) & (l^{(1)}m^{(2)}) & \cdots & (l^{(r)}m^{(2)}) \\
\cos^2\theta\,(m^{(1)}m^{(3)}) & \cos^2\theta\,(m^{(2)}m^{(3)}) & \cdots & \cos^2\theta\,(m^{(3)}m^{(s)}) & (l^{(1)}m^{(3)}) & \cdots & (l^{(r)}m^{(3)}) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cos^2\theta\,(m^{(1)}m^{(s)}) & \cos^2\theta\,(m^{(2)}m^{(s)}) & \cdots & \cos^2\theta & (l^{(1)}m^{(s)}) & \cdots & (l^{(r)}m^{(s)}) \\
(l^{(1)}m^{(1)}) & (l^{(1)}m^{(2)}) & \cdots & (l^{(1)}m^{(s)}) & 1 & \cdots & (l^{(r)}l^{(1)}) \\
(l^{(2)}m^{(1)}) & (l^{(2)}m^{(2)}) & \cdots & (l^{(2)}m^{(s)}) & (l^{(1)}l^{(2)}) & \cdots & (l^{(r)}l^{(2)}) \\
(l^{(3)}m^{(1)}) & (l^{(3)}m^{(2)}) & \cdots & (l^{(3)}m^{(s)}) & (l^{(1)}l^{(3)}) & \cdots & (l^{(r)}l^{(3)}) \\
\cdots & \cdots & & \cdots & \cdots & & \cdots \\
\cdots & \cdots & & \cdots & \cdots & & \cdots \\
(l^{(r)}m^{(1)}) & (l^{(r)}m^{(2)}) & \cdots & (l^{(r)}m^{(s)}) & (l^{(r)}l^{(1)}) & \cdots &
\end{vmatrix}=0.
$$

This is a determinant of the $(r+s)$th order and gives an equation of the sth order in $\cos^2\theta$. Hence θ has s different values. Let θ_1, θ_2, θ_3, ... θ_s be these s values of θ.

Changing cosines into sines in the above equations we may obtain the results in a simple form :—

The constant term then becomes $\left[l^{(1)} \; l^{(2)} \; \ldots \; l^{(r)} \; m^{(1)} \; m^{(2)} \; \ldots \; m^{(s)} \right]^2$ and the coefficient of $\sin^{2s}\theta$ becomes

$$\left[l^{(1)} \; l^{(2)} \; l^{(3)} \ldots \; l^{(r)} \right]^2 . \left[m^{(1)} \; m^{(2)} \; m^{(3)} \ldots \; m^{(s)} \right]^2$$

$$\therefore \operatorname{Sin}^2\theta_1 . \sin^2\theta_2 \; \ldots \; \sin^2\theta_s = \frac{\left[l^{(1)} \; l^{(2)} \ldots \; l^{(r)} \; m^{(1)} \; m^{(2)} \ldots \; m^{(s)} \right]^2}{\left[l^{(1)} \; l^{(2)} \ldots \; l^{(r)} \right]^2 \left[m^{(1)} \; m^{(2)} \ldots \; m^{(s)} \right]^2}$$

Note : In the two determinants (I) and (II) above for determining the values of θ, it is seen that the degree in $\cos^2\theta$ in (I) is r and that in (II) is s. In (I) $\cos^{2(r-s)}\theta$ comes out as a factor.

$\therefore$ $(r-s)$ roots of that equation are zero, *i.e.* there are $r-s$ angles which are all right angles; that is to say, there are $(r-s)$ mutually orthogonal lines in the r-space which are also orthogonal to the s-space. For the r-space and the s-space may be contained in a space of $(r+s)$ dimensions.*

The space orthogonal complementary† to the s-space is an r-space. But in an $(r+s)$-space, two r-spaces always intersect in an $\{r+r-(r+s)\}$ or $(r-s)$-space. In this $(r-s)$-space, through any point can be drawn $(r-s)$ lines, and only $(r-s)$, which are mutually orthogonal. These $(r-s)$ mutually orthogonal lines lie in the given r-space and are all orthogonal to the s-space, being in the orthogonal r-space. These $(r-s)$ orthogonal lines may be taken for the limiting lines in the r-space, but they are orthogonal to all lines in the s-space and consequently to the limiting lines in that space. Thus $(r-s)$ of the minimum angles are right angles.

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* *Vide*—Eugenio Bertini—Introduzione alla Geom. Proe. degli Iperspazi. Cap° I, No. 10.

† Explained in § 28.

Since these satisfy the definition of minimum angles we must include them in the number of minimum angles. In fact they are neither minimum nor maximum. We thus generalise the theorem that the number of minimum angles between an r-space and a s-space is r, of which $(r-s)$ are right angles. Properly speaking there are only s minimum angles.

Cor. :—If in particular we put $r=3$, $s=2$, and θ_1, θ_2 be the minimum angles, (p, q, r) the lines defining the 3-space and (l, m) those defining the plane, the equation reduces to: —*

$$\begin{vmatrix} \cos^2\theta & \cos^2\theta(lm) & (lp) & (lq) & (lr) \\ (lm)\cos^2\theta & \cos^2\theta & (mp) & (mq) & (mr) \\ (lp) & (mp) & 1 & (pq) & (pr) \\ (lq) & (mq) & (pq) & 1 & (qr) \\ (lr) & (mr) & (pr) & (qr) & 1 \end{vmatrix} = 0 \quad \ldots \text{(III)}$$

If we change the cosines into sines, we obtain the result in a simpler form:—Thus

$$\sin^2\theta_1 . \sin^2\theta_2 = [lmpqr]^2/[lm]^2[pqr]^2 \quad \ldots \quad \ldots \text{(IV)}$$

§. 27. To find the minimum angles between two r-spaces. $(2r \ngtr n)$.

Let the r-spaces be defined as in the preceding Article by $l_i^{(j)}$ and $m_i^{(j)}$ $(j=1, 2, 3, \ldots r;\ i=1, 2, 3, \ldots n)$.

Then by the method of §. 26, we obtain a determinant of the $2rth$ order, which yields an equation of the rth order in $\cos^2\theta$.

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* Compare Veronese,—Grunzüge der Geometrie, &c. § 138, Defn. II and Bem. II.

The constant term in the equation may be written in our notation as :—

$[l^{(1)} l^{(2)} \dots l^{(r)} / m^{(1)} m^{(2)} \dots m^{(r)}]^2$ and the coefficient of $\cos^{2r}\theta$

becomes $[l^{(1)} l^{(2)} \dots l^{(r)}]^2 [m^{(1)} m^{(2)} \dots m^{(r)}]^2$.

Thus, if $\theta_1, \theta_2, \theta_3, \dots \theta_r$ be the r minimum angles, we have

$$\cos^2\theta_1 . \cos^2\theta_2 \dots \cos^2\theta_r = [l^{(1)} l^{(2)} \dots l^{(r)} / m^{(1)} m^{(2)} \dots m^{(r)}]^2$$

$$\div [l^{(1)} l^{(2)} \dots l^{(r)}]^2 [m^{(1)} m^{(2)} \dots m^{(r)}]^2 \qquad \dots \text{(I)}$$

Changing cosines into sines we obtain

$$\sin^2\theta_1 . \sin^2\theta_2 \dots \sin^2\theta_r = [l^{(1)} l^{(2)} \dots l^{(r)} m^{(1)} m^{(2)} \dots m^{(r)}]^2$$

$$\div [l^{(1)} l^{(2)} \dots l^{(r)}]^2 [m^{(1)} m^{(2)} \dots m^{(r)}]^2 \qquad \dots \text{(II)}$$

Cor. 1 :—If we put $r=2$, the result becomes

$$\cos^2\theta_1 . \cos^2\theta_2 = [l^{(1)} l^{(2)} / m^{(1)} m^{(2)}]^2 / [l^{(1)} l^{(2)}]^2 [m^{(1)} m^{(2)}]^2$$

which agrees with the result already obtained in §. 23.

$$\text{Also, } \sin^2\theta_1 . \sin^2\theta_2 = [l^{(1)} l^{(2)} m^{(1)} m^{(2)}]^2$$

$$\div [l^{(1)} l^{(2)}]^2 [m^{(1)} m^{(2)}]^2.$$

Cor. 2 :—Putting $r=3$, we obtain

$$\cos^2\theta_1 . \cos^2\theta_2 . \cos^2\theta_3 = [l^{(1)} l^{(2)} l^{(3)} / m^{(1)} m^{(2)} m^{(3)}]^2$$

$$\div [l^{(1)} l^{(2)} l^{(3)}]^2 [m^{(1)} m^{(2)} m^{(3)}]^2.$$

$$\sin^2\theta_1 . \sin^2\theta_2 . \sin^2\theta_3 = [l^{(1)} l^{(2)} l^{(3)} m^{(1)} m^{(2)} m^{(3)}]^2 .$$

$$\div [l^{(1)} l^{(2)} l^{(3)}]^2 [m^{(1)} m^{(2)} m^{(3)}]^2$$

§. **28. Orthogonal Hyper-spaces.** We have seen (§. 10) that through any point of an r-space lying in an n-space ($n>r$), there can be drawn $(n-r)$ lines, and only $(n-r)$, mutually orthogonal and such that each is perpendicular to all lines in the r-space. These are then $(n-r)$ mutually orthogonal lines *normal* to the given r-space. They determine an $(n-r)$-space, such that any line in this is perpendicular to any line in the r-space. This $(n-r)$-space is called the "*orthogonal* Hyper-space" to the given r-space. This may also be termed "the *orthogonal complementary* space" to the given r-space.

§. 29. To show that the minimum angles between any two planes in an n-space are the same as those between the spaces orthogonal to them.

An $(n-2)$-space is orthogonal complementary to a plane. Let the planes be defined by the lines (l, m) and (p, q) respectively all drawn through the origin.

Let λ_i and μ_i ($i=1, 2, 3, \ldots n$) be any two lines, one in each of the spaces orthogonal to the planes.

If θ be the angle between these two lines, we have

$$\cos\theta=\Sigma\lambda_i\,\mu_i \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

Again, since λ lies in the space orthogonal to the plane (l, m) it is perpendicular to all lines in the plane.

$$\text{Consequently, } \sum_{i=1}^{i=n}\lambda_i l_i=0 \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\text{and } \sum_{i=1}^{i=n}\lambda_i m_i=0 \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\text{Similarly, } \sum_{i=1}^{i=n}\mu_i p_i=0 \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$\text{and } \sum_{i=1}^{i=n}\mu_i q_i=0 \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$\text{Also } 1=\sum_{i=1}^{i=n}\lambda_i^2 \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$\text{and } 1=\sum_{i=1}^{i=n}\mu_i^2 \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

Differentiating these 7 equations we obtain

$$0=\sum_{i=1}^{i=n}\lambda_i\ \delta\mu_i+\sum_{i=1}^{i=n}\mu_i\ \delta\lambda_i \quad \dots \quad \dots \quad (1')$$

$$0=\sum_{i=1}^{i=n}l_i\ \delta\lambda_i \quad \dots \quad \dots \quad \dots \quad (2')$$

$$0=\sum_{i=1}^{i=n}m_i\ \delta\lambda_i \quad \dots \quad \dots \quad \dots \quad (3')$$

$$0=\sum_{i=1}^{i=n}p_i\ \delta\mu_i \quad \dots \quad \dots \quad \dots \quad (4')$$

$$0=\sum_{i=1}^{i=n}q_i\ \delta\mu_i \quad \dots \quad \dots \quad \dots \quad (5')$$

$$0=\sum_{i=1}^{i=n}\lambda_i\ \delta\lambda_i \quad \dots \quad \dots \quad \dots \quad (6')$$

$$0=\sum_{i=1}^{i=n}\mu_i\ \delta\mu_i \quad \dots \quad \dots \quad \dots \quad (7')$$

Keeping λ fixed and varying μ, we have $\delta\lambda_1=\delta\lambda_2=\dots=\delta\lambda_n=0$.

From (1′), (4′), (5′) and (7′), we must have a relation of the form—

$$A\lambda_i+B\mu_i+Cp_i+Dq_i=0$$

$$(i=1, 2, 3, \dots n) \quad \dots \quad \dots \quad (A)$$

where A, B, &c., are indeterminate.

Similarly, by keeping μ fixed, we obtain from (1′), (2′), (3′) and (6′)

$$A'\lambda_i+B'\mu_i+C'l_i+D'm_i=0$$

$$(i=1, 2, 3, \dots n) \quad \dots \quad \dots \quad (B)$$

where A', B', &c., are indeterminate.

Multiplying (A) respectively by μ_1, μ_2, … μ_n in order and adding we get, in virtue of the relations (4) and (5)

$$A\cos\theta+B=0 \quad \dots \quad \dots \quad \dots \quad (8)$$

Similarly, from (B), multiplying by λ_1, λ_2, … λ_n and adding we get

$$A'+B'\cos\theta=0 \quad \dots \quad \dots \quad \dots \quad (9)$$

Substituting in (A) and (B) from equations (8) and (9) we obtain equations of the form—

$$\lambda_i - \mu_i \cos\theta = Pp_i + Qq_i \qquad \dots \qquad \dots \qquad \dots \quad (\alpha)$$

and $$\mu_i - \lambda_i \cos\theta = P'l_i + Q'm_i \qquad \dots \qquad \dots \qquad \dots \quad (\beta)$$

$$(i=1, 2, 3, \dots n)$$

Multiplying (α) by $(l_1, l_2, \dots l_n)$ in order and adding we get

$$-(\mu l)\cos\theta = P\,(lp) + Q\,(lq) \qquad \dots \qquad \dots \quad (10)$$

Similarly, multiplying (α) by $m_1, m_2, \dots m_n$ and adding we obtain

$$-(\mu m)\cos\theta = P\,(mp) + Q\,(mq) \qquad \dots \qquad \dots \quad (11)$$

In the same way we obtain from (β)

$$(\mu l) = P' + Q'\,(lm) \qquad \dots \qquad \dots \qquad \dots \quad (12)$$

$$(\mu m) = P'\,(lm) + Q' \qquad \dots \qquad \dots \qquad \dots \quad (13)$$

From these equations (10), (11), (12) and (13), by eliminating (μl) and (μm) we obtain

$$\{P' + Q'(lm)\}\cos\theta + P(lp) + Q'(lq) = 0 \qquad \dots \qquad \dots \quad (14)$$

$$\{P'(lm) + Q'\}\cos\theta + P(mp) + Q(mq) = 0 \qquad \dots \qquad \dots \quad (15)$$

In a similar manner from (α) and (β), after multiplying by $p_1, p_2, \dots p_n$ and $q_1, q_2, \dots q_n$ in order and adding we get

$$\{P + Q(pq)\}\cos\theta + P'(lp) + Q'(mp) = 0 \qquad \dots \qquad \dots \quad (16)$$

$$\{P(pq) + Q\}\cos\theta + P'(lq) + Q'(mq) = 0 \qquad \dots \qquad \dots \quad (17)$$

Eliminating P, Q, P', Q' from equations (14)—(17) we obtain :—

$$\begin{vmatrix} \cos\theta & (lm)\cos\theta & (lp) & (lq) \\ (lm)\cos\theta & \cos\theta & (mp) & (mq) \\ (lp) & (mp) & \cos\theta & \cos\theta(pq) \\ (lq) & (mq) & \cos\theta(pq) & \cos\theta \end{vmatrix} = 0$$

or $$\begin{vmatrix} \cos^2\theta & \cos^2\theta(lm) & (lp) & (lq) \\ \cos^2\theta(lm) & \cos^2\theta & (mp) & (mq) \\ (lp) & (mp) & 1 & (pq) \\ (lq) & (mq) & (pq) & 1 \end{vmatrix} = 0$$

This is the same determinant equation as was obtained in §. **23,** and therefore gives the same values of θ.

Thus *the minimum angles between any two planes are the same as those between two spaces orthogonal to them.*

Note: Here it is seen that the number of minimum angles between two $(n-2)$-spaces is only two; but in general there are $(n-2)$ minimum angles. The reason for this is that the other $(n-4)$ minimum angles all vanish, in view of the fact that the two $(n-2)$-spaces in an n-space are not "independent," but they must intersect in a space of $(n-4)$ dimensions.

§. 30. To show that the minimum angles between any two Hyperspaces are the same as those between spaces orthogonal to them.

Take any two spaces of r and s dimensions respectively $(r+s \ngtr n)$, so that the spaces orthogonal to them are respectively of $(n-r)$ and $(n-s)$ dimensions.

Let the spaces be defined by lines drawn through the origin, whose direction-cosines are those given in §. 26.

Take two lines λ and μ respectively in the two orthogonal spaces. If θ be the angle between these lines, we have—

$$\cos\theta = \sum_{i=1}^{i=n} \lambda_i \mu_i \qquad \ldots \qquad \ldots \qquad (1)$$

$$1 = \sum_{i=1}^{i=n} \lambda_i^2 \qquad \ldots \qquad \ldots \qquad (2)$$

$$1 = \sum_{i=1}^{i=n} \mu_i^2 \qquad \ldots \qquad \ldots \qquad (3)$$

and since the line λ is perpendicular to the r lines of the r-space,

$$0 = \sum_{i=1}^{i=n} \lambda_i l_i^{(j)}$$

$$(j=1, 2, 3, \ldots r) \qquad \ldots \qquad (A)$$

and since μ is perpendicular to the lines of the s-space we have

$$0 = \sum_{i=1}^{i=n} \mu_i m_i^{(t)}$$

$$(t=1, 2, 3, \ldots s) \qquad \ldots \qquad (B)$$

Differentiating the above equations we obtain

$$0 = \sum_{i=1}^{i=n} \lambda_i \delta\mu_i + \sum_{i=1}^{i=n} \mu_i \delta\lambda_i \qquad \ldots \qquad (4)$$

$$0 = \sum_{i=1}^{i=n} \lambda_i \delta\lambda_i \qquad \ldots \qquad \ldots \qquad (5)$$

$$0 = \sum_{i=1}^{i=n} \mu_i \delta\mu_i \qquad \ldots \qquad \ldots \qquad (6)$$

$$0 = \sum_{i=1}^{i=n} l_i^{(j)} \delta\lambda_i \qquad \ldots \qquad \ldots \qquad (7)$$

$$(j = 1, 2, 3. \ldots r)$$

$$0 = \sum_{i=1}^{i=n} m_i^{(t)} \delta\mu_i \qquad \ldots \qquad \ldots \qquad (8)$$

$$(t = 1, 2, 3, \ldots s)$$

Putting $\delta\lambda_1 = \delta\lambda_2 = \ldots = \delta\lambda_n = 0$, we obtain relations of the form $A\lambda_i + B\mu_i + \sum_{t=1}^{t=s} a_t m_i^{(t)} = 0$ ... (9)

$$(i = 1, 2\ 3, \ldots n)$$

where A, B, a_1, a_2, &c. are indeterminate.

Similarly, putting $\delta\mu_1 = \delta\mu_2 = \ldots = \delta\mu_n = 0$, we get

$$A'\lambda_i + B'\mu_i + \sum_{j=1}^{j=r} b_j l_i^{(j)} = 0 \qquad \ldots \qquad \ldots \qquad (10)$$

$$(i = 1, 2, 3, \ldots n)$$

Multiplying (9) by $\mu_1\ \mu_2, \ldots \mu_n$ in order and adding we obtain, in virtue of the relations (B)—

$$A \cos\theta + B = 0$$

Similarly, we obtain $A' + B' \cos\theta = 0$

Thus, equations (9) may be written as

$$\lambda_i - \mu_i \cos\theta = \sum_{t=1}^{t=s} a_t m_i^{(t)} \qquad \ldots \qquad (\alpha)$$

$$(i = 1, 2, 3, \ldots n)$$

and equations (10) may be written as

$$\mu_i - \lambda_i \cos\theta = \sum_{j=1}^{j=r} \beta_j l_i^{(j)} \qquad \ldots \qquad \ldots \qquad (\beta)$$

$$(i = 1, 2, 3, \ldots n)$$

Multiplying (α) respectively by $\overset{(j)}{l_i}$ ($i = 1, 2, 3, \ldots n$; $j = 1, 2, 3, \ldots r$) in order and adding we obtain, in virtue of the relations (A),—

$$\left(\mu\,\overset{(j)}{l}\right)\cos\theta = \sum_{t=1}^{t=s}\alpha_t\left(\overset{(j)}{l}\,\overset{(t)}{m}\right) \qquad \ldots \quad (\gamma)$$

$$(j = 1, 2, 3, \ldots r)$$

From (β) we obtain, in a similar manner.

$$\left(\mu\,\overset{(j)}{l}\right) = \sum_{p=1}^{p=r}\beta_p\left(\overset{(p)}{l}\,\overset{(j)}{l}\right) \quad \ldots \qquad \ldots \quad (\delta)$$

$$(j = 1, 2, 3, \ldots r.)$$

Eliminating $\left(\mu\,\overset{(j)}{l}\right)$ between (γ) and (δ) we obtain

$$\sum_{t=1}^{t=s}\alpha_t\left(\overset{(j)}{l}\,\overset{(t)}{m}\right) + \cos\theta\sum_{p=1}^{p=r}\beta_p\left(\overset{(p)}{l}\,\overset{(j)}{l}\right)=0$$

$$(j = 1, 2, 3, \ldots r) \ldots \quad (\mathrm{A}')$$

Multiplying (α) and (β) respectively by $\overset{(t)}{m_i}$ ($t=1, 2, 3 \ldots s$; $i=1, 2, 3 \ldots n$) in order and proceeding in the same way, we obtain

$$\sum_{j=1}^{j=r}\beta_j\left(\overset{(j)}{l}\,\overset{(t)}{m}\right) + \cos\theta\sum_{q=1}^{q=s}\alpha_q\left(\overset{(q)}{m}\,\overset{(t)}{m}\right) = 0$$

$$(t = 1, 2, 3, \ldots s.) \qquad \ldots \quad (\mathrm{B}')$$

Eliminating ($\alpha_1, \alpha_2 \ldots \alpha_s$) and ($\beta_1, \beta_2 \ldots \beta_r$) from these ($r+s$) equations in (A′) and (B′) we obtain the same determinant as was obtained in §26.

Hence the values of θ are the same as in the case of two Hyper-spaces of r and s dimensions.

N.B. Here the number of minimum angles is s; but the orthogonal spaces being of (n-r) and (n-s) dimensions ($r>s$), the number of minimum angles between them should generally be (n-r). The reason for this is that the remaining (n-r-s) minimum angles all vanish, in view of the fact that two Hyper-spaces of (n-r) and (n-s) dimensions in an n-space cannot be "independent," but must intersect in a lower space of (n-r-s) dimensions. Thus we get $\{(n\text{-}r) - (n\text{-}r\text{-}s)\}$ or s minimum angles.

§ 31. To show that the minimum angles between any two planes in an n-space are complements of the minimum angles between any of them and the space orthogonal to the other.

Let the planes be defined by the lines (l, m) and (p, q) respectively.

Take any line (L) in the $(n\text{-}2)$-space orthogonal to the plane (l, m) and another line (P) in the plane (p, q).

Since (L) lies in the space orthogonal to the plane (l, m) we must have

$$\sum_{i=1}^{i=n} \mathrm{L}_i\, l_i = 0 \qquad \ldots \qquad \ldots \qquad (1)$$

$$\sum_{i=1}^{i=n} \mathrm{L}_i\, m_i = 0 \qquad \ldots \qquad \ldots \qquad (2)$$

Also,

$$\sum_{i=1}^{i=n} L_i{}^2 = 1 \qquad \ldots \qquad \ldots \qquad (3)$$

And, since P lies in the plane (p, q), we may take

$$\mathrm{P}_i = \lambda p_i + \mu q_i, \qquad \ldots \qquad \ldots \qquad (4)$$

$$(i = 1, 2, 3, \ldots n)$$

$$\sum_{i=1}^{i=n} \mathrm{P}_i^2 = 1 = \lambda^2 + \mu^2 + 2\lambda\mu\, (\overset{\frown}{pq}) \qquad \ldots \qquad (5)$$

If ϕ be the angle between (L) and (P), we have

$$\operatorname{Cos} \phi = \sum_{i=1}^{i=n} \mathrm{L}_i\, (\lambda p_i + \mu q_i)$$

$$= \lambda\, (\mathrm{L}p) + \mu\, (\mathrm{L}q) \qquad \ldots \qquad \ldots \qquad (6)$$

Differentiating equations (1), (2), (3), (5) & (6) we obtain

$$o = \sum_{i=1}^{i=n} l_i\, \delta \mathrm{L}_i \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7)$$

$$o = \sum_{i=1}^{i=n} m_i\, \delta \mathrm{L}_i \qquad \ldots \qquad \ldots \qquad \ldots \qquad (8)$$

$$o = \sum_{i=1}^{i=n} \mathrm{L}_i\, \delta \mathrm{L}_i \qquad \ldots \qquad \ldots \qquad \ldots \qquad (9)$$

$$o = \delta\lambda\,\{\lambda+\mu(pq)\}+\delta\mu\,\{\lambda(pq)+\mu\} \quad \dots \quad \dots \quad (10)$$

$$o = (Lp)\,\delta\lambda + (Lq)\delta\mu+\lambda \sum_{i=1}^{i=n} p_i\,\delta L_i + \mu \sum_{i=1}^{i=n} q_i\,\delta L_i$$

$$= (Lp)\,\delta\lambda+ (Lq)\delta\mu+ \sum_{i=1}^{i=n} (\lambda p_i+\mu q_i)\delta L_i, \quad \dots \quad (11)$$

Keeping (L) fixed, we have

$$\left.\begin{array}{l} k\,(Lp) = \lambda+\mu\,(pq) \\ k\,(Lq) = \lambda\,(pq)+\mu \end{array}\right\} \text{ where k is indeterminate.}$$

Multiplying these by λ and μ respectively and adding we obtain $k \cos\phi=1$, $\therefore\ k=1/\cos\phi$.

$$\therefore\ (Lp)=\cos\phi\,\{\lambda+\mu\,(pq)\} \quad \dots \quad \dots \quad (12)$$

$$(Lq)=\cos\phi\,\{\lambda(pq)+\mu\} \quad \dots \quad \dots \quad (13)$$

Keeping (P) fixed we obtain

$$al_i+bm_i+cL_i+(\lambda p_i+\mu q_i)=o.$$
$$(i=1, 2, 3, \ \dots\ n.) \quad \dots \quad (14)$$

where a, b, & c are indeterminate.

Multiplying (14) successively by l_i, m_i, L_i, p_i, q_i $(i=1, 2, \dots n)$ in order and adding we get

$$\left.\begin{array}{r} a+b\,(lm)+\lambda\,(lp)+\mu(lq)=o \\ a\,(lm)+b+\lambda\,(mp)+\mu\,(mq)=o \\ c+\lambda\,(Lp)+\mu\,(Lq)=o \\ a\,(lp)+b\,(mp)+c\,(Lp)+\lambda+\mu\,(pq)=o \\ a\,(lq)+b\,(mq)+c\,(Lq)+\lambda\,(pq)+\mu=o \end{array}\right\} \quad \dots \quad [A)$$

By (12) & (13), $c= -\cos\phi$ and

$$c\,(Lp)= -\cos^2\phi\{\lambda+\mu(pq)\}$$

$$c\,(Lq)= -\cos^2\phi\{\lambda(pq)+\mu\}$$

$\therefore$ These equations (A) can be written after eliminating c, (Lp), (Lq) as follow :—

$$\left.\begin{array}{r} \lambda\,\mathrm{Sin}^2\,\phi + \mu\,\mathrm{Sin}^2\,\phi\,(pq) + a\,(lp)+b\,(mp) = o \\ \lambda\,\mathrm{Sin}^2\,\phi\,(pq) + \mu\,\mathrm{Sin}^2\,\phi + a\,(lq) + b\,(mq) = o \\ \lambda\,(lp) + \mu\,(lq) + a + b\,(lm) = o \\ \lambda\,(mp) + \mu\,(mq) + a\,(lm) + b = o \end{array}\right\} \quad (15)$$

Eliminating λ, μ, a, b from these equations we obtain the following determinant equation :—

$$\begin{vmatrix} \text{Sin}^2\,\phi & \text{Sin}^2\,\phi\,(pq) & (lp) & (mp) \\ \text{Sin}^2\,\phi\,(pq) & \text{Sin}^2\,\phi & (lq) & (mq) \\ (lp) & (lq) & 1 & (lm) \\ (mp) & (mq) & (lm) & 1 \end{vmatrix} = o \quad \ldots \quad (16)$$

If ϕ_1 and ϕ_2 are the values of ϕ, we have

$$\text{Sin}^2\phi_1\ \text{Sin}^2\phi_2 = [lm/pq]^2 \Big/ [lm]^2\ [pq]^2.$$

But the expression on the right-hand side is the same as was obtained in §23 for $\text{Cos}^2\theta_1 . \text{Cos}^2\theta_2$. Further if we replace ϕ by $(\frac{\pi}{2} - \phi)$ in (16) we obtain the equation of §23.

$\therefore$ θ_1 and θ_2 are respectively the complements of ϕ_1 and ϕ_2.

Again, from the symmetry in the result, we infer at once that ϕ_1 and ϕ_2 are also the minimum angles between the plane (l, m) and the space orthogonal to the plane (p, q).

Cor :—If we put $\text{Sin}^2\phi = 1 - \text{Cos}^2\phi$, then

$$\text{Cos}^2\phi_1 . \text{Cos}^2\phi_2 = [lm\underset{\frown}{pq}]^2 \Big/ [lm]^2\ [pq]^2$$

$$= \text{Sin}^2\theta_1 . \text{Sin}^2\theta_2 . \quad (\S 23).$$

§ 32. To find the angles between a s-space and a space orthogonal to a given r-space ($r > s$ and $r + s \not> n$).

Let the r-space and the s-space be defined as in §26.

Take a line (L) in the space orthogonal to the r-space, and another line (P) in the s-space.

Then we have $$\sum_{i=1}^{i=n} L_i\, l_i^{(j)} = o$$

$$(j = 1, 2, 3, \ldots r) \quad \ldots \quad \ldots \quad (1)$$

and $$\sum_{i=1}^{i=n} L_i^2 = 1 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Again, since (P) lies in the s-space, we may take

$$P_i = \sum_{t=1}^{t=s} \lambda_t\, m_i^{(t)} \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

where λ_t is indeterminate and $i = 1, 2, 3, \ldots n$.

also $$\sum_{i=1}^{i=n} P_i^2 = 1$$

$$= \sum_{j=1}^{j=s} \sum_{t=1}^{t=s} \lambda_j \lambda_t \left(m^{(} \quad m^{(t)} \right) \qquad \ldots \qquad \ldots \quad (4)$$

$$= \sum_1^s \lambda^2{}_t + 2 \sum \lambda_a \lambda_b \left(m^{(a)} m^{(b)} \right).$$

where $a = 1, 2, 3, \ldots s$; $b = a + 1, a + 2, \ldots s$.

If ϕ be the angle between (L) and (P) we have

$$\cos \phi = \sum_{i=1}^{i=n} L_i\, P_i = \sum_{i=1}^{i=n} L_i \left(\lambda_1 m_i^{(1)} + \lambda_2 m_i^{(2)} + \ldots + \lambda_s m_i^{(s)}\right)$$

$$= \sum_{t=1}^{t=s} \lambda_t \left(L\, m^{(t)}\right) \qquad \ldots \qquad \ldots \quad (5)$$

Differentiating equations (1), (2), (4) and (5) we obtain

$$o = \sum_{i=1}^{i=n} l_i^{(j)}\, \delta L_i \qquad \ldots \qquad \ldots \qquad \ldots \quad (6)$$

$$(j = 1, 2, 3, \ldots\ldots r)$$

$$o = \sum_{i=1}^{i=n} L_i\, \delta L_i \qquad \ldots \qquad \ldots \qquad \ldots \quad (7)$$

$$o = \sum_{p=1}^{p=s} \delta \lambda_p \sum_{t=1}^{t=s} \lambda_t \left(m^{(p)} m^{(t)}\right) \qquad \ldots \qquad \ldots \quad (8)$$

and $$o = \sum_{t=1}^{t=s} \delta \lambda_t \left(L\, m^{(t)}\right) + \sum_{i=1}^{i=n} \delta L_i \sum_{t=1}^{t=s} \lambda_t\, m_i^{(t)}. \qquad \ldots \quad (9)$$

Keeping (L) fixed we obtain

$$k\,(L\,m^{(p)}) = \sum_{t=1}^{t=s} \lambda_t\,(m^{(p)}\,m^{(t)})$$

$(p=1, 2, 3, \ldots s)$,

where k is in determinate.

Multiplying these by $\lambda_1, \lambda_2, \ldots \lambda_s$ successively and adding we get $k \cos\phi = 1$; $\therefore\ k = 1 / \cos\phi$.

$$\therefore (L\,m^{(p)}) = \cos\phi \left\{ \sum_{i=1}^{t=s} \lambda_t\,(m^{(p)}\,m^{(t)}) \right\} . \qquad \ldots \quad (10)$$

$(p=1, 2, 3, \ldots s)$

Again, keeping (P) fixed we obtain

$$\sum_{j=1}^{j=r} a_j\, l_i^{(j)} + b\,L_i + \sum_{t=1}^{t=s} \lambda_t\, m_i^{(t)} = o \qquad \ldots \quad (11)$$

$(i = 1, 2, 3, \ldots n)$

Multiplying equations (11) by $L_1, L_2, .\ L_n$ respectively and adding we get $b + \cos\phi = o \ \therefore b = -\cos\phi$. ... (12)

Again, multiplying the same equations successively by

$l_i^{(j)}$ and $m_i^{(t)}$ $(j=1, 2, 3, \ldots r;\ t=1, 2, 3, \cdots s;\ i=1, 2, 3, \ldots n)$ in order and adding we get

$$\sum_{j=1}^{j=r} a_j\,(l^{(j)}\,l^{(q)}) + \sum_{t=1}^{t=s} \lambda_t\,(m^{(t)}\,l^{(q)}) = o \qquad \ldots \quad (13)$$

$(q=1, 2, 3, \ldots r.)$

and $\displaystyle\sum_{j=1}^{j=r} a_j\,(l^{(j)}\,m^{(t)}) - \cos\phi\,(L\,m^{(t)}) + \frac{(L\,m^{(t)})}{\cos\phi} = o$

$$i.e.\ \sum_{j=1}^{j=r} a_j\, l^{(j)}\,m^{(t)}) + \sin^2\phi\,(L\,m^{(t)}) / \cos\phi = o \ \ldots \quad \ldots \quad (14)$$

$(t = 1, 2, 3, \ldots\ldots s)$

Eliminating $(L\ m^{(t)})$ between (10) and (14) we get equations.

Thus altogether we obtain $(r+s)$ equations from (13) and (14) involving the $(r+s)$ quantities $a_1, a_2, \ldots a_r$ and $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_s$.

Eliminating them we get the following determinant equation:—

$$\begin{vmatrix}
\sin^2\phi & \sin^2\phi(\overset{(1)}{m}\,\overset{(2)}{m}) & \cdots & \sin^2\phi(\overset{(1)}{m}\,\overset{(s)}{m}) & (\overset{(1)}{m}\,\overset{(1)}{l}) & (\overset{(1)}{m}\,\overset{(2)}{l}) & \cdots & (\overset{(1)}{m}\,\overset{(r)}{l}) \\
\sin^2\phi(\overset{(1)}{m}\,\overset{(2)}{m}) & \sin^2\phi & \cdots & \sin^2\phi(\overset{(2)}{m}\,\overset{(s)}{m}) & (\overset{(2)}{m}\,\overset{(1)}{l}) & (\overset{(2)}{m}\,\overset{(2)}{l}) & \cdots & (\overset{(2)}{m}\,\overset{(r)}{l}) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\sin^2\phi(\overset{(1)}{m}\,\overset{(s)}{m}) & \sin^2\phi(\overset{(2)}{m}\,\overset{(s)}{m}) & \cdots & \sin^2\phi & (\overset{(s)}{m}\,\overset{(1)}{l}) & (\overset{(s)}{m}\,\overset{(2)}{l}) & \cdots & (\overset{(s)}{m}\,\overset{(r)}{l}) \\
(\overset{(1)}{l}\,\overset{(1)}{m}) & (\overset{(1)}{l}\,\overset{(2)}{m}) & \cdots & (\overset{(1)}{l}\,\overset{(s)}{m}) & 1 & (\overset{(1)}{l}\,\overset{(2)}{l}) & \cdots & (\overset{(1)}{l}\,\overset{(r)}{l}) \\
(\overset{(2)}{l}\,\overset{(1)}{m}) & (\overset{(2)}{l}\,\overset{(2)}{m}) & \cdots & (\overset{(2)}{l}\,\overset{(s)}{m}) & (\overset{(2)}{l}\,\overset{(1)}{l}) & 1 & \cdots & (\overset{(2)}{l}\,\overset{(r)}{l}) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
(\overset{(r)}{l}\,\overset{(1)}{m}) & (\overset{(r)}{l}\,\overset{(2)}{m}) & \cdots & (\overset{(r)}{l}\,\overset{(s)}{m}) & (\overset{(r)}{l}\,\overset{(1)}{l}) & (\overset{(r)}{l}\,\overset{(2)}{l}) & \cdots & 1
\end{vmatrix} = 0$$

This is the same determinant to find Sin ϕ as was obtained in § **26 (II)** to find cos θ and gives s values of ϕ—(ϕ_1, $\phi_2 \ldots \phi_s$).

Hence the ϕ's in this equation are the complements of θ's in § **26**.

If we put $\mathrm{Sin}^2\phi = 1 - \cos^2\phi$, we obtain

$$\mathrm{Cos}^2\phi_1 \,.\, \cos^2\phi_2 \ldots \cos^2\phi_s = \frac{\left[\overset{(1)}{l}\ \overset{(2)}{l}\ \ldots \overset{(r)}{l}\ \overset{(1)}{m}\ \overset{(2)}{m}\ \ldots \overset{(s)}{m}\right]^2}{\left[\overset{(1)}{l}\ \overset{(2)}{l}\ \ldots \overset{(r)}{l}\right]^2 \left[\overset{(1)}{m}\ \overset{(2)}{m}\ \ldots \overset{(s)}{m}\right]^2}$$

$$= \mathrm{Sin}^2\theta_1 \,.\, \mathrm{Sin}^2\theta_2 \ldots \mathrm{Sin}^2\theta_s \,.\, (\S 26, \text{cor}).$$

Note.—From the symmetry in the above determinant it is seen that we should obtain the same result if we calculate the angles between the r-space and the space orthogonal to the s-space. In this case we obtain the determinant (I) of § 26, and the remark made in the corollary of that article applies to this case also.

Thus we may generalise the theorem in the following form:—

"*If there are any two spaces of r and s dimensions respectively* $(r + s \ngtr n)$, *the angles between these spaces are complements of those between any of them and the space orthogonal to the other.*"

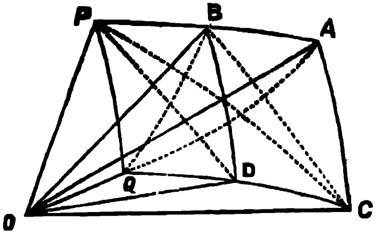
§ 33. Minimal lines:

In defining angles between any two spaces we have taken one line in each of the spaces and then determined the minimum angles between these two lines. These lines in their limiting positions *i.e.*, when the angle between them is a minimum, will be called "*minimal* lines." The two lines in each limiting position will be called "corresponding lines," and when the lines are drawn through the same point, the plane determined by them will be called a "*minimal* plane."

§ 34. The minimal lines in each of two given planes are at right angles to each other.

Let AOB and COD be any two planes drawn through any origin O. Since five points will suffice to determine the planes, they can be drawn so as to lie in a Hyper-space of four dimensions. Let OA and OB be the minimal lines in one plane and OC, OD in the other, so that the angles AOC and BOD are minimum angles and the planes AOC and BOD are "minimal planes." It is required to show that OB and OD are respectively perpendicular to OA and OC; and consequently the plane BOD is absolutely perpendicular to AOC.

Since ∠AOC is a minimum angle between the planes AOB and COD, the plane AOC is a common perpendicular plane to AOB and COD.



∴ ∠A and ∠C are right angles. For similar reasons, ∠B and ∠D are right angles. Thus the four lines OA, OB, OC, OD are such that ∠ˢA, B, C, D are right angles.

It is required to shew that ∠AOB and ∠COD are right angles. If not, let AOP and COQ be each a right angle. Then POQ is a plane which intersects both the planes AOB and COD and ∠AOP and ∠COQ are right angles.

Now planes AOC, COQ, AOQ lie in a 3-space determined by OA, OC, OQ; and OC is perpendicular to OQ and ∠C is a right angle. ∴ ∠AOQ is a right angle.*

Thus OA is perpendicular to both the lines OP and OQ; and consequently to the plane POQ. ∴ ∠P is a right angle. Similarly ∠Q is also a right angle.

∴ The plane POQ is a common perpendicular plane to AOB and COD *i.e.*, POQ is a minimal plane. But there are *only* two minimal planes. ∴ The plane POQ must coincide with BOD ∴ ∠AOB and ∠COD are right angles. Consequently the minimal planes are absolutely perpendicular to each other.

Cor. I: If the two given planes are absolutely perpendicular, ∠ˢAOC, BOD are both right angles. Hence the four lines OA, OB, OC, OD are mutually orthogonal, as also the four planes AOB, AOC, BOC, COD. Hence the two planes AOD and BOC are also orthogonal to each other, as well as to the four planes. Thus the four lines determine a system of mutually orthogonal lines.

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* *Vide*—Todhunter, Sph. Trigonometry, §73.

Cor. II: If the two planes have a common point, any plane through this point and cutting the two minimal planes in two right lines will be perpendicular to both. For the lines of intersection are mutually orthogonal, as also are the angles ∠C and ∠D right angles. Thus the minimal planes (absolutely perpendicular to each other) have an infinite number of common perpendicular planes, on which they cut out right angles.

§ **35.** An analytical proof of the above theorem may be given as follows :—

Let the planes be defined as in §23.

Let A_1, A_2 and B_1, B_2 be the two values of A and B respectively in equations 8—11 (§23), corresponding to the limiting positions of the minimal lines. Also let C_1, C_2 and D_1, D_2 be the corresponding values of C and D respectively in the other plane.

Now the angle between the minimal lines corresponding to the two values of C and D being ϕ, we have

$$\text{Cos}\,\phi = \sum_{i=1}^{i=n} (C_1 m_i + D_1 m'_i)(C_2 m_i + D_2 m'_i)$$

$$= C_1C_2 + (C_1D_2 + D_1C_2)(mm') + D_1D_2. \qquad \ldots \quad (1)$$

The equations determining the values of A, B, C, D are

$$\left.\begin{array}{l} 1 = A^2 + B^2 + 2AB(ll') \\ 1 = C^2 + D^2 + 2\,CD\,(mm') \end{array}\right\} \qquad \ldots \quad (A)$$

and

$$\left.\begin{array}{l} A\cos\theta + B\cos\theta\ (ll') - C\ (lm) - D\ (lm') = 0 \\ A\cos\theta\ (ll') + B\cos\theta - C\ (l'm) - D\ (l'm') = 0 \\ A\ (lm) + B\ (l'm) - C\cos\theta - D\cos\theta\ (mm') = 0 \\ A\ (lm') + B\ (l'm') - C\ (mm')\cos\theta - D\cos\theta = 0 \end{array}\right\} \quad (B)$$

Solving the first two equations in (B) for A and B we obtain

$$\frac{A}{C\{(lm)-(l'm)\,(ll')\}+D\{(lm')-(l'm')\,(ll')\}}$$

$$=\frac{B}{C\{\,(l'm)-(lm)\ (ll')\,\}+D\{\,(l'm')\ (ll')\,\}}$$

$$=\frac{1}{\cos\theta\,[ll']^2}$$

From the last two equations in (B) we have

$$\frac{\mathrm{C}+\mathrm{D}(mm')}{\mathrm{C}(mm')+\mathrm{D}}=\frac{\mathrm{A}(lm)+\mathrm{B}(l'm)}{\mathrm{A}(lm')+\mathrm{B}(l'm')}$$

Substituting the values of A and B in this, we obtain $\frac{\mathrm{C}+\mathrm{D}(mm')}{\mathrm{C}(mm')+\mathrm{D}}=\frac{\Omega}{\Psi}$, where

$$\Omega\equiv\mathrm{C}\{(lm)^2+(l'm)^2-2(lm)(l'm)(ll')\}+\mathrm{D}\{(lm)(lm')$$
$$+(l'm)(l'm')-(ll')(lm)(l'm')-(ll')(lm')(l'm)\}.$$

$$\Psi\equiv\mathrm{C}\{(lm)(lm')+(l'm)(l'm')-(l'm)(lm')(ll')$$
$$-(ll')(lm)(l'm')\}+\mathrm{D}\{(lm')^2+(l'm')^2$$
$$-2(ll')(lm')(l'm')\}.$$

Put $\alpha\equiv(lm)^2+(l'm)^2-2(ll')(lm)(l'm)$.

$\beta\equiv(lm')^2+(l'm')^2-2(ll')(lm')(l'm')$.

$\gamma\equiv(lm)(lm')+(l'm)(l'm')-(l'm)(lm')(ll')$
$-(ll')(lm)(l'm')$.

Then we have

$$\frac{\mathrm{C}+\mathrm{D}(mm')}{\mathrm{C}(mm')+\mathrm{D}}=\frac{\mathrm{C}\cdot\alpha+\mathrm{D}\cdot\gamma}{\mathrm{C}\cdot\gamma+\mathrm{D}\cdot\beta}$$

or $\{\mathrm{C}+\mathrm{D}(mm')\}\{\mathrm{C}\gamma+\mathrm{D}\beta\}=\{\mathrm{C}(mm')+\mathrm{D}\}\{\mathrm{C}\alpha+\mathrm{D}\gamma\}$

or $\mathrm{C}^2\{\gamma-\alpha(mm')\}+\mathrm{CD}\{\gamma(mm')+\beta-\gamma(mm')-\alpha\}$
$+\mathrm{D}^2\{\beta(mm')-\gamma\}=0.$

or $\mathrm{C}^2\{\gamma-\alpha(mm')\}+\mathrm{CD}\{\beta-\alpha\}+\mathrm{D}^2\{\beta(mm')-\gamma\}=0.$

This is a quadratic in C/D and hence gives two values of C/D, *namely*, $\mathrm{C}_1/\mathrm{D}_1$ and $\mathrm{C}_2/\mathrm{D}_2$.

$$\left.\begin{aligned}&\text{Then, }\frac{\mathrm{C}_1\mathrm{C}_2}{\mathrm{D}_1\mathrm{D}_2}=\frac{\beta(mm')-\gamma}{\gamma-\alpha(mm')}\\&\text{and }\frac{\mathrm{C}_1}{\mathrm{D}_1}+\frac{\mathrm{C}_2}{\mathrm{D}_2}=-\frac{\beta-\alpha}{\gamma-\alpha(mm')}\end{aligned}\right\}\quad\ldots\quad\ldots\quad(2)$$

From (1) we have $\cos\phi=\mathrm{C}_1\mathrm{C}_2+(\mathrm{C}_1\mathrm{D}_2+\mathrm{C}_2\mathrm{D}_1)(mm')+\mathrm{D}_1\mathrm{D}_2$

$$=\mathrm{D}_1\mathrm{D}_2\left\{\frac{\mathrm{C}_1\mathrm{C}_2}{\mathrm{D}_1\mathrm{D}_2}+\left(\frac{\mathrm{C}_1}{\mathrm{D}_1}+\frac{\mathrm{C}_2}{\mathrm{D}_2}\right)(mm')+1\right\}$$

Substituting the values from (2) we obtain

$$\cos\phi=\mathrm{D}_1\mathrm{D}_2\left\{\frac{\beta\,(mm')-\gamma}{\gamma-a\,(mm')}-\frac{(\beta-a)\,(mm')}{\gamma-a\,(mm')}+1\right\}$$

$$=\frac{\mathrm{D}_1\mathrm{D}_2}{\gamma-a\,(mm')}\left\{\beta(mm')-\gamma-\beta(mm')+a(mm')+\gamma-a\,(mm')\right\}=0$$

$\therefore\phi=2n\pi\pm\frac{\pi}{2}$. *i.e.* the two minimal lines OC and OD are mutually perpendicular.

Similarly, the two corresponding lines in the other plane are mutually perpendicular.

Cor : The minimal planes are also mutually orthogonal.

§36. To express the minimum angles between any two planes in an n-space in terms of the mutual distances between the generating points.

Let the planes be defined by two sets of 3 given points whose coordinates are $\overset{(j)}{x}_i$ and $\overset{(j)}{y}_i$ $(j=1, 2, 3\ ;\ i=1, 2, 3\ldots n)$ respectively.

If θ_1 and θ_2 be the minimum angles between the planes, we have by §, 24,

$$\cos^2\theta_1.\cos^2\theta_2=\frac{[\Sigma(1\,x_1\,x_2)\,/\,(1\,y_1\,y_2)]^2}{\Sigma(1\,x_1\,x_2)^2\,\Sigma(1\,y_1\,y_2)^2}\ \cdots\qquad(1)$$

But we have

$$\begin{vmatrix} 1 & 0 & \ldots & 0 & 0 \\ \Sigma\left(\overset{(1)}{x}_i\right)^2 & -2\overset{(1)}{x}_1 & \ldots & -2\overset{(1)}{x}_n & 1 \\ \Sigma\left(\overset{(2)}{x}_i\right)^2 & -2\overset{(2)}{x}_1 & \ldots & -2\overset{(2)}{x}_n & 1 \\ \Sigma\left(\overset{(3)}{x}_i\right)^2 & -2\overset{(3)}{x}_1 & \ldots & -2\overset{(3)}{x}_n & 1 \end{vmatrix}\times\begin{vmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & \overset{(1)}{y}_1 & \ldots & \overset{(1)}{y}_n & \Sigma\left(\overset{(1)}{y}_i\right)^2 \\ 1 & \overset{(2)}{y}_1 & \ldots & \overset{(2)}{y}_n & \Sigma\left(\overset{(2)}{y}_i\right)^2 \\ 1 & \overset{(3)}{y}_1 & \ldots & \overset{(3)}{y}_n & \Sigma\left(\overset{(3)}{y}_i\right)^2 \end{vmatrix}$$

$$=4\ \Sigma\begin{vmatrix} 1 & \overset{(1)}{x}_1 & \overset{(1)}{x}_2 \\ 1 & \overset{(2)}{x}_1 & \overset{(2)}{x}_2 \\ 1 & \overset{(3)}{x} & \overset{(3)}{x}_2 \end{vmatrix}\qquad\begin{vmatrix} 1 & \overset{(1)}{y}_1 & \overset{(1)}{y}_2 \\ 1 & \overset{(2)}{y}_1 & \overset{(2)}{y}_2 \\ 1 & \overset{(3)}{y}_1 & \overset{(3)}{y}_2 \end{vmatrix}$$

$$\equiv 4\,\Sigma\,(1\,x_1\,x_2)\,/\,(1\,y_1\,y_2)$$

$$=\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & \Big(\overset{(1)}{x}\ \overset{(1)}{y}\Big)^2 & \Big(\overset{(2)}{x}\ \overset{(1)}{y}\Big)^2 & \Big(\overset{(3)}{x}\ \overset{(1)}{y}\Big)^2 \\ 1 & \Big(\overset{(1)}{x}\ \overset{(2)}{y}\Big)^2 & \Big(\overset{(2)}{x}\ \overset{(2)}{y}\Big)^2 & \Big(\overset{(3)}{x}\ \overset{(2)}{y}\Big)^2 \\ 1 & \Big(\overset{(1)}{x}\ \overset{(3)}{y}\Big)^2 & \Big(\overset{(2)}{x}\ \overset{(3)}{y}\Big)^2 & \Big(\overset{(2)}{x}\ \overset{(3)}{y}\Big)^2 \end{vmatrix} \equiv \Delta_{xy} \text{ (say)} \quad \ldots \quad (2)$$

where $(\overset{(1)}{x}\ \overset{(2)}{y})$ stands for the distance between the points denoted by $\overset{(1)}{x}$ and $\overset{(1)}{y}$.

$$\therefore \quad \text{The numerator of } (1) = \frac{1}{4^2}\Delta^2_{xy} \qquad \ldots \qquad \ldots \quad (3)$$

$$\text{Again, } \Sigma(1\ x_1\ x_2)^2 = \Sigma \begin{vmatrix} 1 & \overset{(1)}{x}_1 & \overset{(1)}{x}_2 \\ 1 & \overset{(2)}{x}_1 & \overset{(2)}{x}_2 \\ 1 & \overset{(3)}{x}_1 & \overset{(3)}{x}_2 \end{vmatrix}^2 = \Sigma \begin{vmatrix} 1 & \overset{(1)}{x}_1 & \overset{(1)}{x}_2 \\ 1 & \overset{(2)}{x}_1 & \overset{(2)}{x}_2 \\ 1 & \overset{(3)}{x}_1 & \overset{(3)}{x}_2 \end{vmatrix} \begin{vmatrix} 1 & \overset{(1)}{x}_1 & \overset{(1)}{x}_2 \\ 1 & \overset{(2)}{x}_1 & \overset{(2)}{x}_2 \\ 1 & \overset{(3)}{x}_1 & \overset{(3)}{x}_2 \end{vmatrix}$$

$$= \frac{1}{4}\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \Big(\overset{(1)}{x}\ \overset{(2)}{x}\Big)^2 & \Big(\overset{(1)}{x}\ \overset{(3)}{x}\Big)^2 \\ 1 & \Big(\overset{(2)}{x}\ \overset{(1)}{x}\Big)^2 & 0 & \Big(\overset{(2)}{x}\ \overset{(3)}{x}\Big)^2 \\ 1 & \Big(\overset{(3)}{x}\ \overset{(1)}{x}\Big)^2 & \Big(\overset{(2)}{x}\ \overset{(3)}{x}\Big)^2 & 0 \end{vmatrix}$$

$$\equiv \frac{1}{4}\Delta_{xx} \text{ (say).} \qquad \ldots. \qquad \ldots. \qquad \ldots. \quad (4)$$

$$\text{Similarly, } \Sigma\,(1\ y_1\ y_2)^2 = \tfrac{1}{4}\Delta_{yy} \qquad \ldots \qquad \ldots \quad (5)$$

$$\therefore \quad \cos^2\theta_1 . \cos^2\theta_2 = \tfrac{1}{16}\Delta^2_{xy} \div \tfrac{1}{4}\Delta_{xx} . \tfrac{1}{4}\Delta_{yy}$$

$$= \Delta^2_{xy} \big/ \Delta_{xx}\Delta_{yy}.$$

where Δ_{xy}, Δ_{xx}, Δ_{yy} stand respectively for the determinants in (2), (4) and (5).

Again, by the same §24,

$$\operatorname{Sin}^2\theta_1 . \sin^2\theta_2 = \left[\frac{\overset{(1)}{x}\ \overset{(2)}{x}\ \overset{(3)}{x}\ \overset{(1)}{y}\ \overset{(2)}{y}\ \overset{(3)}{y}}{0,1.}\right]^2$$

$$\div\ \tfrac{1}{4}\Delta_{xx}\cdot\tfrac{1}{4}\Delta_{yy} \qquad \ldots \qquad (6)$$

The numerator in (6) can be expressed in terms of the mutual distances between the points as follows :—

$$\begin{vmatrix}
1 & 0 & \ldots & 0 & 0 & 0 \\
\Sigma(\overset{(1)}{x}_i)^2 & -2\overset{(1)}{x}_1 & \ldots & -2\overset{(1)}{x}_n & 0 & 1 \\
\Sigma(\overset{(2)}{x}_i)^2 & -2\overset{(2)}{x}_1 & \ldots & -2\overset{(2)}{x}_n & 0 & 1 \\
\Sigma(\overset{(3)}{x}_i)^2 & -2\overset{(3)}{x}_1 & \ldots & -2\overset{(3)}{x}_n & 0 & 1 \\
\Sigma(\overset{(1)}{y}_i)^2 & -2\overset{(1)}{y}_1 & \ldots & -2\overset{(1)}{y}_n & 1 & 1 \\
\Sigma(\overset{(2)}{y}_i)^2 & -2\overset{(2)}{y}_1 & \ldots & -2\overset{(2)}{y}_n & 1 & 1 \\
\Sigma(\overset{(3)}{y}_i)^2 & -2\overset{(3)}{y}_1 & \ldots & -2\overset{(3)}{y}_n & 1 & 1
\end{vmatrix} \times$$

$$\begin{vmatrix}
0 & 0 & \ldots & 0 & 0 & 1 \\
1 & \overset{(1)}{x}_1 & \ldots & \overset{(1)}{x}_n & 0 & \Sigma(\overset{(1)}{x}_i)^2 \\
1 & \overset{(2)}{x}_1 & \ldots & \overset{(2)}{x}_n & 0 & \Sigma(\overset{(2)}{x}_i)^2 \\
1 & \overset{(3)}{x}_1 & \ldots & \overset{(3)}{x}_n & 0 & \Sigma(\overset{(3)}{x}_i)^2 \\
1 & \overset{(1)}{y}_1 & \ldots & \overset{(1)}{y}_n & 1 & \Sigma(\overset{(1)}{y}_i)^2 \\
1 & \overset{(2)}{y}_1 & \ldots & \overset{(2)}{y}_n & 1 & \Sigma(\overset{(2)}{y}_i)^2 \\
1 & \overset{(3)}{y}_1 & \ldots & \overset{(3)}{y}_n & 1 & \Sigma(\overset{(3)}{y}_i)^2
\end{vmatrix}$$

$$\equiv\ 16\left[\frac{\overset{(1)}{x}\ \overset{(2)}{x}\ \overset{(3)}{x}\ \overset{(1)}{y}\ \overset{(2)}{y}\ \overset{(3)}{y}}{0,1}\right]^2$$

The same matrices multiplied give the determinant—

$$\begin{vmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & (\overset{(1)}{x}\ \overset{(2)}{x})^2 & (\overset{(1)}{x}\ \overset{(3)}{x})^2 & (\overset{(1)}{x}\ \overset{(1)}{y})^2 & (\overset{(1)}{x}\ \overset{(2)}{y})^2 & (\overset{(1)}{x}\ \overset{(3)}{y})^2 \\
1 & (\overset{(2)}{x}\ \overset{(1)}{x})^2 & 0 & (\overset{(2)}{x}\ \overset{(3)}{x})^2 & (\overset{(2)}{x}\ \overset{(1)}{y})^2 & (\overset{(2)}{x}\ \overset{(2)}{y})^2 & (\overset{(2)}{x}\ \overset{(3)}{y})^2 \\
1 & (\overset{(3)}{x}\ \overset{(1)}{x})^2 & (\overset{(3)}{x}\ \overset{(2)}{x})^2 & 0 & (\overset{(3)}{x}\ \overset{(1)}{y})^2 & (\overset{(3)}{x}\ \overset{(2)}{y})^2 & (\overset{(3)}{x}\ \overset{(3)}{y})^2 \\
1 & (\overset{(1)}{x}\ \overset{(1)}{y})^2 & (\overset{(2)}{x}\ \overset{(1)}{y})^2 & (\overset{(3)}{x}\ \overset{(1)}{y})^2 & 1 & (\overset{(1)}{y}\ \overset{(2)}{y})^2+1 & (\overset{(1)}{y}\ \overset{(3)}{y})^2+1 \\
1 & (\overset{(1)}{x}\ \overset{(2)}{y})^2 & (\overset{(2)}{x}\ \overset{(2)}{y})^2 & (\overset{(3)}{x}\ \overset{(2)}{y})^2 & (\overset{(1)}{y}\ \overset{(2)}{y})^2+1 & 1 & (\overset{(2)}{y}\ \overset{(3)}{y})^2+1 \\
1 & (\overset{(1)}{x}\ \overset{(3)}{y})^2 & (\overset{(2)}{x}\ \overset{(3)}{y})^2 & (\overset{(3)}{x}\ \overset{(3)}{y})^2 & (\overset{(3)}{y}\ \overset{(1)}{y})^2+1 & (\overset{(2)}{y}\ \overset{(3)}{y})^2+1 & 1
\end{vmatrix}$$

This may again be written as—

$$\begin{vmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & (\overset{(1)}{x}\ \overset{(2)}{x})^2 & (\overset{(1)}{x}\ \overset{(3)}{x})^2 & (\overset{(1)}{x}\ \overset{(1)}{y})^2 & (\overset{(1)}{x}\ \overset{(2)}{y})^2 & (\overset{(1)}{x}\ \overset{(3)}{y})^2 \\
0 & 1 & (\overset{(1)}{x}\ \overset{(2)}{x})^2 & 0 & (\overset{(2)}{x}\ \overset{(3)}{x})^2 & (\overset{(2)}{x}\ \overset{(1)}{y})^2 & (\overset{(2)}{x}\ \overset{(2)}{y})^2 & (\overset{(2)}{x}\ \overset{(3)}{y})^2 \\
0 & 1 & (\overset{(1)}{x}\ \overset{(3)}{x})^2 & (\overset{(2)}{x}\ \overset{(3)}{x})^2 & 0 & (\overset{(3)}{x}\ \overset{(1)}{y})^2 & (\overset{(3)}{x}\ \overset{(2)}{y})^2 & (\overset{(3)}{x}\ \overset{(3)}{y})^2 \\
0 & 1 & (\overset{(1)}{x}\ \overset{(1)}{y})^2 & (\overset{(2)}{x}\ \overset{(1)}{y})^2 & (\overset{(3)}{x}\ \overset{(1)}{y})^2 & 1 & (\overset{(1)}{y}\ \overset{(2)}{y})^2+1 & (\overset{(1)}{y}\ \overset{(3)}{y})^2+1 \\
0 & 1 & (\overset{(1)}{x}\ \overset{(2)}{y})^2 & (\overset{(2)}{x}\ \overset{(2)}{y})^2 & (\overset{(3)}{x}\ \overset{(2)}{y})^2 & (\overset{(1)}{y}\ \overset{(2)}{y})^2+1 & 1 & (\overset{(2)}{y}\ \overset{(3)}{y})^2+1 \\
0 & 1 & (\overset{(1)}{x}\ \overset{(3)}{y})^2 & (\overset{(2)}{x}\ \overset{(3)}{y})^2 & (\overset{(3)}{x}\ \overset{(3)}{y})^2 & (\overset{(1)}{y}\ \overset{(3)}{y})^2+1 & (\overset{(2)}{y}\ \overset{(3)}{y})^2+1 & 1
\end{vmatrix}$$

Adding the 1st row to each of the 6th, 7th and 8th rows and then subtracting the 2nd row, we obtain (denoting the points $\overset{(1)}{x}, \overset{(2)}{x}, \overset{(3)}{x}$ by the numerals 1, 2, 3 and $\overset{(1)}{y}, \overset{(2)}{y}, \overset{(3)}{y}$ by 1′, 2′, 3′— :

$$\begin{vmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & (2,1)^2 & (3,1)^2 & (1',1)^2 & (2',1)^2 & (3',1)^2 \\ 0 & 1 & (1,2)^2 & 0 & (3,2)^2 & (1',2)^2 & (2',2)^2 & (3',2)^2 \\ 0 & 1 & (1,3)^2 & (2,3)^2 & 0 & (1',3)^2 & (2',3)^2 & (3',3)^2 \\ 1 & 1 & (1,1')^2 & (2,1')^2 & (3,1')^2 & 0 & (2',1')^2 & (3',1')^2 \\ 1 & 1 & (1,2')^2 & (2,2')^2 & (3,2')^2 & (1',2')^2 & 0 & (2',3')^2 \\ 1 & 1 & (1,3')^2 & (2,3')^2 & (3,3')^2 & (1',3')^2 & (2,'3')^2 & 0 \end{vmatrix}$$

$$\equiv \Delta^{123}_{1'2'3'} \quad \text{(say).}$$

$$\therefore \left[\frac{1231'2'3'}{0,1}\right]^2 = \tfrac{1}{16}\Delta^{123}_{1'2'3'}.$$

$$\therefore \text{Sin}^2\theta_1.\text{Sin}^2\theta_2 = \Delta^{123}_{1'2'3'} \div \Delta_{xx}\ \Delta_{xy}.$$

§37. To prove that the orthogonal projection of a plane circle on another plane is an ellipse, and that the square of its area is equal to the sum of the squares of the areas of the ellipses of projection on the coordinate planes determined by each pair of axes.

Let (l,m) be the plane of the circle, the origin its centre and let (p,q) be any other plane.

If θ_1, θ_2 be the minimum angles between these planes, we have by **§23**, $\text{Cos}\theta_1.\text{Cos}\theta_2 = [lm/pq]/[lm]\ [pq]$.

We have also seen that the minimal lines in each plane are mutually orthogonal. Therefore if we take the two minimal lines for diameters of the circle, they will project into two orthogonal lines through the projection of the centre.

Hence two conjugate diameters of the circle will project into two perpendicular diameters of the projected figure, *i.e*, into two axes.

The projection is a conic* and since the projecting lines all meet the plane of projection in finite points, the projection will be a closed figure, and consequently it is an ellipse.

If a be the radius of the circle, the product of the axes of the ellipse$=a\cos\theta_1 . a\cos\theta_2$.

$$=a^2\cos\theta_1.\cos\theta_2$$

$$=a^2\,[lm/pq]/[lm]\,[pq]$$

$\therefore$ The area of the ellipse $=\pi a^2[lm/pq]/[lm]\,[pq]$

If the plane (p,q) be any of the coordinate planes (say) the plane $(1,2)$,

then the area$=\pi a^2\begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix}\div\left[lm\right]$

$\therefore$ Sum of the squares of the areas of projection

$$=\pi^2a^4\sum\begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix}^2\div[lm]^2=\pi^2a^4[lm]^2/[lm]^2$$

$$=\pi^2a^4=\text{ square of the area of the circle.}$$

§38. New definition of angles between two planes:

From the form of the expression for $\cos\theta_1 . \cos\theta_2$ between two planes (§24) we may readily infer a new definition of angles between two planes. For, in (4) of §24, we find that the numerator is the sum of the products of the type

$$\begin{vmatrix} 1 & \overset{(1)}{x_1} & \overset{(1)}{x_2} \\ 1 & \overset{(2)}{x_1} & \overset{(2)}{x_2} \\ 1 & \overset{(3)}{x_1} & \overset{(3)}{x_2} \end{vmatrix}\times\begin{vmatrix} 1 & \overset{(1)}{y_1} & \overset{(1)}{y_2} \\ 1 & \overset{(2)}{y_1} & \overset{(2)}{y_2} \\ 1 & \overset{(3)}{y_1} & \overset{(3)}{y_2} \end{vmatrix}\quad\ldots\ (1)$$

Now, the first factor $=2\times$ area of the triangle in the axial plane (1,2), which is the projection of the triangle S determined by the three given points $\overset{(1)}{x}, \overset{(2)}{x}, \overset{(3)}{x}$.

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* This follows from §4, Cap. XII, Bertini—loc. cit.

Similarly, the second factor = 2 × area of the triangle in the axial plane (1,2), which is the projection of the triangle S′ in the plane determined by the three points $\overset{(1)}{y}, \overset{(2)}{y}, \overset{(3)}{y}$.

If S_{12} and S'_{12} be these areas, the numerator becomes $4\Sigma S_{12}S'_{12}$. Again, we have seen §22 that $S^2 = \Sigma S^2_{rs}$,

$$(r=1, 2, \ldots n\ ;\ s=r+1, r+2\ldots n)$$

$$\therefore\ \Sigma \begin{vmatrix} 1 & \overset{(1)}{x_1} & \overset{(1)}{x_2} \\ 1 & \overset{(2)}{x_1} & \overset{(2)}{x_2} \\ 1 & \overset{(3)}{x_1} & \overset{(3(}{x_2} \end{vmatrix}^2 = 4S^2 \text{ and } \Sigma \begin{vmatrix} 1 & \overset{(1)}{y_1} & \overset{(1)}{y_2} \\ 1 & \overset{(2)}{y_1} & \overset{(2)}{y_2} \\ 1 & \overset{(3)}{y_1} & \overset{(3)}{y_2} \end{vmatrix}^2 = 4S'^2$$

Hence the expression (4) of §24 becomes

$$\cos^2\theta_1 \cdot \cos^2\theta_2 = (4\Sigma S_{rs}\ S'_{rs})^2/4S^2 \cdot 4S'^2$$

$$= (\Sigma S_{rs}\ S'_{rs})^2/S^2\ S'^2$$

$$\therefore\quad SS' \cos\theta_1 \cdot \cos\theta_2 = \Sigma S_{rs}\ S'_{rs}$$

$$(r=1, 2, 3\ldots n\ ;\ S=r+1, r+2, r+3, \ldots n)$$

Since any plane area can be divided into an infinite number of small triangles, the reasoning will apply when S and S′ are any two plane areas.

Thus if S and S′ are any two plane areas in the given planes respectively and S_{12}, S'_{12} etc. are respectively their projections on the axial planes, then the angles between the planes of S and S′ are given by

$$SS' \cos\theta_1 \cdot \cos\theta_2 = \Sigma S_{rs}\ S'_{rs}$$

$$[r=1, 2, 3, \ldots n\ ;\ s=r+1, r+2,\ r+3, \ldots n]\ \ldots\quad (2)$$

§39. To find the shortest distance between two given planes in an n-space.

The shortest distance between any two planes is the intercept on the line which meets both of them orthogonally. Hence it is the projection of the line joining any two points of the planes on a common perpendicular line to both the planes.

The planes are each determined by three given points. Thus they can be drawn so as to lie in a 5-space determined by the six points.* Again, the line on which the shortest distance lies intersects both the planes, and thus having two of its points in the same 5-space, entirely lies in it. Hence the problem reduces to one in 5-dimensional Geometry.

Now, let (x_i, y_i, z_i) and (x_i', y_i', z_i') be two sets of 3 points defining the planes $(i=1, 2, 3, 4, 5)$.

Let the direction-cosines of the lines joining x to y and z be given respectively by $l_i^{(j)}$ and those of the lines joining x' to y' and z' be given by $m_i^{(j)}$ $(j=1, 2; i=1, 2, 3, 4, 5)$.

Let λ be the line on which the shortest distance lies, so that its direction-cosines are λ_i $(i=1, 2, 3, 4, 5)$.

Now, the content V_5 of the "join" of the six given points is given by

$$(5!V_5)^2\dagger = \begin{vmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & y_1 & y_2 & y_3 & y_4 & y_5 \\ 1 & z_1 & z_2 & z_3 & z_4 & z_5 \\ 1 & x_1' & x_2' & x_3' & x_4' & x_5' \\ 1 & y_1' & y_2' & y_3' & y_4' & y_5' \\ 1 & z_1' & z_2' & z_3' & z_4' & z_5' \end{vmatrix}^2$$

$$= \begin{vmatrix} 0 & x_1-x_1' & x_2-x_2' & x_3-x_3' & x_4-x_4' & x_5-x_5' \\ 0 & x_1-y_1 & x_2-y_2 & x_3-y_3 & x_4-y_4 & x_5-y_5 \\ 0 & x_1-z_1 & x_2-z_2 & x_3-z_3 & x_4-z_4 & x_5-z_5 \\ 1 & x_1' & x_2' & x_3' & x_4' & x_5' \\ 0 & x_1'-y_1' & x_2'-y_2' & x_3'-y_3' & x_4'-y_4' & x_5'-y_5' \\ 0 & x_1'-z_1' & x_2'-z_2' & x_3'-z_3' & x_4'-z_4' & x_5'-z_5' \end{vmatrix}^2$$

$$= \begin{vmatrix} x_1-x_1' & x_2-x_2' & x_3-x_3' & x_4-x_4' & x_5-x_5' \\ x_1-y_1 & x_2-y_2 & x_3-y_3 & x_4-y_4 & x_5-y_5 \\ x_1-z_1 & x_2-z_2 & x_3-z_3 & x_4-z_4 & x_5-z_5 \\ x_1'-y_1' & x_2'-y_2' & x_3'-y_3' & x_4'-y_4' & x_5'-y_5' \\ x_1'-z_1' & x_2'-z_2' & x_3'-z_3' & x_4'-z_4' & x_5'-z_5' \end{vmatrix}^2 \dots (1)$$

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* *Vide*—Bertini—Introduzione etc., Cap I, No. 10-11.

† *Vide*—Proc. of the London Math. Soc. Vols. XVIII and XIX.

If $\overline{xy}=\rho$, $\overline{xz}=\sigma$; $\overline{x'y'}=\rho'$, $\overline{x'z'}=\sigma'$, where $\overline{xy}$ stands for the distance between the points x and y, then we have

$$\left.\begin{array}{l}
\dfrac{x_1-y_1}{\overset{(1)}{l_1}}=\dfrac{x_2-y_2}{\overset{(1)}{l_2}}=\ldots=\dfrac{x_5-y_5}{\overset{(1)}{l_5}}=\rho, \\[2ex]
\dfrac{x_1-z_1}{\overset{(2)}{l_1}}=\dfrac{x_2-z_2}{\overset{(2)}{l_2}}=\ldots=\dfrac{x_5-z_5}{\overset{(2)}{l_5}}=\sigma; \\[2ex]
\dfrac{x_1'-y_1'}{\overset{(1)}{m_1}}=\dfrac{x_2'-y_2'}{\overset{(1)}{m_2}}=\ldots=\dfrac{x_5'-y_5'}{\overset{(1)}{m_5}}=\rho', \\[2ex]
\dfrac{x_1'-z_1'}{\overset{(2)}{m_1}}=\dfrac{x_2'-z_2'}{\overset{(2)}{m_2}}=\ldots=\dfrac{x_5'-z_5'}{\overset{(2)}{m_5}}=\sigma'.
\end{array}\right\}$$

∴ Substituting these values in the determinant (1) we obtain

$$5!V_5=\begin{vmatrix}
x_1-x_1 & x_2-x_2 & x_3-x_3 & x_4-x_4 & x_5-x_5 \\
\overset{(1)}{l_1}\rho & \overset{(1)}{l_2}\rho & \overset{(1)}{l_3}\rho & \overset{(1)}{l_4}\rho & \overset{(1)}{l_5}\rho \\
\overset{(2)}{l_1}\sigma & \overset{(2)}{l_2}\sigma & \overset{(2)}{l_3}\sigma & \overset{(2)}{l_4}\sigma & \overset{(2)}{l_5}\sigma \\
\overset{(1)}{m_1}\rho' & \overset{(1)}{m_2}\rho' & \overset{(1)}{m_3}\rho' & \overset{(1)}{m_4}\rho' & \overset{(1)}{m_5}\rho' \\
\overset{(2)}{m_1}\sigma' & \overset{(2)}{m_2}\sigma' & \overset{(2)}{m_3}\sigma' & \overset{(2)}{m_4}\sigma' & \overset{(2)}{m_5}\sigma'
\end{vmatrix}$$

$$\therefore\ V_5=\frac{\rho\rho'\sigma\sigma'}{5!}\left\{\left(x_1\overset{(1)}{l_2}\overset{(2)}{l_3}\overset{(1)}{m_4}\overset{(2)}{m_5}\right)-\left(x'_1\overset{(1)}{l_2}\overset{(2)}{l_3}\overset{(1)}{m_4}\overset{(2)}{m_5}\right)\right\}\ \ldots(2)$$

Now, the line λ is perpendicular to both the planes and consequently to all lines in them.

∴ By the condition of perpendicularity we obtain :—

$$\left.\begin{array}{l}
\sum\limits_{i=1}^{i=5}\lambda_i\overset{(1)}{l_i}=0,\quad \sum\limits_{i=1}^{i=5}\lambda_i\overset{(2)}{l_i}=0 \\[3ex]
\sum\limits_{i=1}^{i=5}\lambda_i\overset{(1)}{m_i}=0,\quad \sum\limits_{i=1}^{i=5}\lambda_i\overset{(2)}{m_i}=0
\end{array}\right\}\ \ldots\ (3)$$

Solving these equations for $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, we obtain

$$\frac{\lambda_1}{(\overset{(1)}{l_2}\,\overset{(2)}{l_3}\,\overset{(1)}{m_4}\,\overset{(2)}{m_5})} = \frac{\lambda_2}{(\overset{(1)}{l_1}\,\overset{(2)}{l_3}\,\overset{(1)}{m_4}\,\overset{(2)}{m_5})} = \frac{\lambda_3}{(\overset{(1)}{l_1}\,\overset{(2)}{l_2}\,\overset{(1)}{m_4}\,\overset{(2)}{m_5})} = \frac{\lambda_4}{(\overset{(1)}{l_1}\,\overset{(2)}{l_2}\,\overset{(1)}{m_3}\,\overset{(2)}{m_5})}$$

$$= \frac{\lambda_5}{(\overset{(1)}{l_1}\,\overset{(2)}{l_2}\,\overset{(1)}{m_3}\,\overset{(2)}{m_4})} = \frac{1}{[\overset{(1)}{l}\,\overset{(2)}{l}\,\overset{(1)}{m}\,\overset{(2)}{m}]} \qquad \dots \quad (4)$$

for $\sum\limits_{i=1}^{i=5} \lambda_i^2 = 1$.

If δ denotes the shortest distance (s. d.) between the planes, it is the projection of $\overline{xx'}$ on the line (λ), and consequently

$$\delta = (x_1 - x'_1)\,\lambda_1 + (x_2 - x'_2)\,\lambda_2 + (x_3 - x'_3)\,\lambda_3 + (x_4 - x'_4)\lambda_4 + (x_5 - x'_5)\lambda_5.$$

Substituting these values of λ's from (4) in this we obtain

$$\delta\,[\overset{(1)}{l}\,\overset{(2)}{l}\,\overset{(1)}{m}\,\overset{(2)}{m}] = \begin{vmatrix} x_1 - x'_1 & x_2 - x'_2 & x_3 - x'_3 & x_4 - x'_4 & x_5 - x'_5 \\ \overset{(1)}{l_1} & \overset{(1)}{l_2} & \overset{(1)}{l_3} & \overset{(1)}{l_4} & \overset{(1)}{l_5} \\ \overset{(2)}{l_1} & \overset{(2)}{l_2} & \overset{(2)}{l_3} & \overset{(2)}{l_4} & \overset{(2)}{l_5} \\ \overset{(1)}{m_1} & \overset{(1)}{m_2} & \overset{(1)}{m_3} & \overset{(1)}{m_4} & \overset{(1)}{m_5} \\ \overset{(2)}{m_1} & \overset{(2)}{m_2} & \overset{(2)}{m_3} & \overset{(2)}{m_4} & \overset{(2)}{m_5} \end{vmatrix}$$

$$= 5!\,V_5/\rho\rho'\sigma\sigma'.$$

or $$V_5 = \frac{\rho\rho'\sigma\sigma'}{5!}.\delta\,[\overset{(1)}{l}\,\overset{(2)}{l}\,\overset{(1)}{m}\,\overset{(2)}{m}] \qquad \dots \qquad \dots \qquad (5)$$

Now the area Δ of the triangle $xyz = \frac{1}{2}\rho\sigma \sin\theta$, where θ is the angle between the lines $\overline{xy}$ and $\overline{xz}$.

or $\quad \Delta = \frac{1}{2}\rho\sigma\,[\overset{(1)}{l}\,\overset{(2)}{l}]$.

Similarly, the area Δ' of the triangle $x'\,y'\,z'$ is given by

$$\Delta = \tfrac{1}{2}\rho'\,\sigma'\ [\overset{(1)}{m}\,\overset{(2)}{m}]$$

$$\therefore\ 4\Delta\Delta' = \rho\rho'\sigma\sigma'\ [\overset{(1)}{l}\,\overset{(2)}{l}]\ [\overset{(1)}{m}\,\overset{(2)}{m}]$$

$$\therefore V_5 = \frac{4\Delta\Delta'}{5!\,[\overset{(1)}{l}\ \overset{(2)}{l}\,]\,[\overset{(1)}{m}\ \overset{(2)}{m}\,]} . \delta\,[\overset{(1)}{l}\ \overset{(2)}{l}\ \overset{(1)}{m}\ \overset{(2)}{m}\,]$$

$$\text{or } (30V_5)^2 = \Delta^2.\Delta'^2.\delta^2\,[\overset{(1)}{l}\ \overset{(2)}{l}\ \overset{(1)}{m}\ \overset{(2)}{m}\,]^2 \,/\, [\overset{(1)}{l}\ \overset{(2)}{l}\,]^2\,[\overset{(1)}{m}\ \overset{(2)}{m}\,]^2$$

$$= \Delta^2.\Delta'^2.\delta^2 \sin^2\theta_1 \cdot \sin^2\theta_2{}^{*} \ (\S.23) \quad \ldots \quad (A)$$

where θ_1 and θ_2 are the minimum angles between the planes.

$$\text{Now, } 4\Delta^2 = \sum \begin{vmatrix} 1 & x_1 & x_2 \\ 1 & y_1 & y_2 \\ 1 & z_1 & z_2 \end{vmatrix}^2 \equiv \sum (1\ xyz)^2$$

$$\text{and } 4\,\Delta'^2 = \sum \begin{vmatrix} 1 & x'_1 & x'_2 \\ 1 & y'_1 & y'_2 \\ 1 & z'_1 & z'_2 \end{vmatrix}^2 \equiv \sum (1\ x'y'z')^2$$

$$\text{and } \sin^2\theta_1 \cdot \sin^2\theta_2 = \left[\frac{xyz\ x'y'z'}{0,1}\right]^2 \div \sum (1\,xyz)^2 \cdot \sum (1x'y'z')^2$$

(§. 24)

$$= \left[\frac{xyz\ x'y'z'}{0,1}\right]^2 \div 4\Delta^2 \cdot 4\Delta'^2.$$

$$\therefore \Delta^2\Delta'^2.\sin^2\theta_1 \cdot \sin^2\theta_2 = \tfrac{1}{16}\left[\frac{xyz\ x'y'z'}{0,1}\right]^2$$

Thus finally, from (A) we obtain

$$(30\ V_5)^2 = \frac{\delta^2}{16}\left[\frac{xyz\ x'y'z'}{0,1}\right]^2$$

$$\text{or } (5!\ V_5)^2 = \delta^2\left[\frac{xyz\ x'y'z'}{0,1}\right]^2$$

$$\therefore \delta^2 = (5!\ V_5)^2 \div \left[\frac{xyz\ x'y'z'}{0,1}\right]^2$$

---

* This exactly corresponds to the formula in 3-dimensional Geometry—*vide* Salmon's Geometry of three dimensions, Ex. 3, p. 50.

$$= \Sigma \begin{vmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & y_1 & y_2 & y_3 & y_4 & y_5 \\ 1 & z_1 & z_2 & z_3 & z_4 & z_5 \\ 1 & x'_1 & x'_2 & x'_3 & x'_4 & x'_5 \\ 1 & y'_1 & y'_2 & y'_3 & y'_4 & y'_5 \\ 1 & z'_1 & z'_2 & z'_3 & z'_4 & z'_5 \end{vmatrix}^2$$

$$\div \Sigma \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ x_1 & y_1 & z_1 & x'_1 & y'_1 & z'_1 \\ x_2 & y_2 & z_2 & x'_2 & y'_2 & z'_2 \\ x_3 & y_3 & z_3 & x'_3 & y'_3 & z'_3 \\ x_4 & y_4 & z_4 & x'_4 & y'_4 & z'_4 \end{vmatrix}^2$$

Thus δ is expressed in terms of the coordinates of the generating points of the planes.

Note.—The method is perfectly general and can be applied in the case of two spaces of any number of dimensions. Thus the shortest distance between two spaces $S_r \left(\overset{(0)}{x}, \overset{(1)}{x}, \ldots \overset{(r)}{x} \right)$ and $S_{r'} \left(\overset{(0)}{y}, \overset{(1)}{y}, \ldots \overset{(r')}{y} \right)$ is given by:—

$$\delta^2 = \left\{ (r+r'+1)!\, V_{r+r'+1} \right\}^2 \div \left[\frac{\overset{(0)}{x} \ldots \overset{(r)}{x} \; \overset{(0)}{y} \ldots \overset{(r')}{y}}{0,\, 1} \right]^2$$

CHAPTER III: Hyper-Spheres or Spherics.

§40. Definitions: An *r-spheric* in an n-space is a surface of the second order of r dimensions. It is the locus of points in an r-space, which are equidistant from a fixed point in the same, called the *centre.*

Any line drawn through the centre to intersect the surface is called a "diameter." Any plane through the centre intersecting the surface is called a "diametral plane"; and in general, any k-space $(k < r)$ drawn through the centre to intersect the r-spheric is called a "diametral k-space."*

§41. To find the equations of the circum-circle of a triangle, the co-ordinates of whose vertices are given.

Let α_i, β_i, γ_i $(i=1, 2, 3, \ldots\ldots n)$ be the three vertices of a triangle. The equation of any n-spheric may be written as:—

$$\sum_{i=1}^{i=n} x_i^2 + 2\sum_{i=1}^{i=n} A_i x_i + C_0 = 0 \qquad \ldots \qquad \ldots \quad (1)$$

where A_i and C_0 are arbitrary constants.

If this passes through the three given points $(\alpha_i, \beta_i, \gamma_i)$, we must have those arbitrary constants connected by the relations:—

$$\sum_{i=1}^{i=n} \alpha_i^2 + 2\sum_{i=1}^{i=n} A_i \alpha_i + C_0 = 0 \qquad \ldots \qquad \ldots \quad (2)$$

$$\sum_{i=1}^{i=n} \beta_i^2 + 2\sum_{i=1}^{i=n} A_i \beta_i + C_0 = 0 \qquad \ldots \qquad \ldots \quad (3)$$

$$\sum_{i=1}^{i=n} \gamma_i^2 + 2\sum_{i=1}^{i=n} A_i \gamma_i + C_0 = 0 \qquad \ldots \qquad \ldots \quad (4)$$

---

*A number of theorems on the intersection of diametral spaces (Diametralraüme) with Spherics (Kugeloberfläche) have been enunciated by Veronese. *Vide*—Gründzuge der Geometrie &c. § 174. We here proceed with the analytical nvestigations of certain properties of general spherics.

Eliminating $A_1, A_2, \ldots\ldots A_n, C_0$ from these equations, we obtain the equations of a set of n-spherics, having a common circle of intersection passing through the three given points,—

i.e. the equations of the circumcircle are

$$\begin{vmatrix} \sum\limits_{i=1}^{i=n} x_i^2 & x_1 & x_2 \ldots\ldots\ldots x_n & 1 \\ \sum\limits_{i=1}^{i=n} \alpha_i^2 & \alpha_1 & \alpha_2 \ldots\ldots\ldots \alpha_n & 1 \\ \sum\limits_{i=1}^{i=n} \beta_i^2 & \beta_1 & \beta_2 \ldots\ldots\ldots \beta_n & 1 \\ \sum\limits_{i=1}^{i=n} \gamma_i^2 & \gamma_1 & \gamma_2 \ldots\ldots\ldots \gamma_n & 1 \end{vmatrix} = 0 \quad \ldots \quad (A)$$

It is possible to choose $A_1, A_2, A_3 \ldots\ldots A_n, C_0$, so that the centre of the n-spheric lies in the plane—

$$\begin{vmatrix} x_1 & x_2 & x_3 \ldots\ldots\ldots x_n & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \ldots\ldots\ldots \alpha_n & 1 \\ \beta_1 & \beta_2 & \beta_3 \ldots\ldots\ldots \beta_n & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 \ldots\ldots\ldots \gamma_n & 1 \end{vmatrix} = 0 \quad \ldots \quad (B)$$

In this case the co-ordinates of the centre of the circum-circle are the same as those of the n-spheric through the three given points.

The co-ordinates of the centre of the n-spheric are $(-A_1, -A_2$ $-A_3, \ldots\ldots -A_n)$ and the radius R is given by

$$R^2 = \sum_{i=1}^{i=n} A_i^2 - C_0.$$

Thus from the equations (2), (3) and (4) and the $(n-2)$ equations in (B), we can determine the co-ordinates of the centre and hence the radius of the n-spheric.

§42. To find the radius of the circum-circle.

Through the three given points, a plane can be drawn and consequently the radius of the n-spheric through the points, with centre in their plane, is the same as that of a plane circle.

Let a, b, c be the lengths of the sides of the triangle. Then, by Plane Trigonometry, we have

$$4\,R^2=\frac{a^2}{sin^2 A},\ \text{where A is the angle at the vertex } (\alpha)\ \ldots \qquad (I)$$

Now, the direction-cosines of the sides are given by

$$\frac{\alpha_i-\beta_i}{c},\quad \frac{\alpha_i-\gamma_i}{b}\ (i=1,\ 2,\ 3\ldots\ldots n)\ \text{respectively.}$$

$$\therefore sin^2 A=\Sigma\begin{vmatrix}\frac{\alpha_1-\beta_1}{c} & \frac{\alpha_2-\beta_2}{c}\\ \frac{\alpha_1-\gamma_1}{b} & \frac{\alpha_2-\gamma_2}{b}\end{vmatrix}^2=\frac{1}{b^2c^2}\Sigma\begin{vmatrix}\alpha_1-\beta_1 & \alpha_2-\beta_2\\ \alpha_1-\gamma_1 & \alpha_2-\gamma_2\end{vmatrix}^2$$

$$=\frac{1}{b^2c^2}\cdot\Sigma\begin{vmatrix}1 & 1 & 1\\ \alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{vmatrix}^2\ \ldots\qquad\ldots\ (II)$$

From (I) we have

$$4\,R^2=\frac{a^2b^2c^2}{\Sigma\begin{vmatrix}1 & 1 & 1\\ \alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{vmatrix}^2}=\frac{\sum\limits_{i=1}^{i=n}(\alpha_i-\beta_i)^2\sum\limits_{i=1}^{i=n}(\beta_i-\gamma_i)^2\sum\limits_{i=1}^{i=n}(\gamma_i-\alpha_i)^2}{\Sigma\begin{vmatrix}1 & 1 & 1\\ \alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{vmatrix}^2}$$

If the lengths of the sides (a, b, c) are given, the result may be stated as :—

$$2R^2=\frac{2a^2b^2c^2}{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}$$

$$=-\begin{vmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{vmatrix} \div \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}.$$

$$\therefore {}^{*}2R^2 \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{vmatrix} = 0 \quad \ldots \text{(III)}$$

§43. To find the co-ordinates of the centre of the circum-circle.

From §41, we have the following $(n+1)$ equations:—

$$\left.\begin{array}{l} \sum\limits_{i=1}^{i=n} \alpha_i^2 + 2\sum\limits_{i=1}^{i=n} A_i\,\alpha_i + C_0 = 0 \\ \\ \sum\limits_{i=1}^{i=n} \beta_i^2 + 2\sum\limits_{i=1}^{i=n} A_i\,\beta_i + C_0 = 0 \\ \\ \sum\limits_{i=1}^{i=n} \gamma_i^2 + 2\sum\limits_{i=1}^{i=n} A_i\,\gamma_i + C_0 = 0 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \; (a)$$

$$\begin{vmatrix} -A_1 & -A_2 & -A_3 \ldots\ldots\ldots -A_n & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \ldots\ldots\ldots\ldots \alpha_n & 1 \\ \beta_1 & \beta_2 & \beta_3 \ldots\ldots\ldots\ldots \beta_n & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 \ldots\ldots\ldots\ldots \gamma_n & 1 \end{vmatrix} = 0 \quad \ldots \; (b)$$

To solve (b) we have

$$-A_i = \lambda\alpha_i + \mu\beta_i + \nu\gamma_i$$

$$(i = 1, 2, 3, \ldots\ldots\ldots n)$$

and $1 = \lambda + \mu + \nu$

---

* Compare—Frost—Solid Geometry §589.

where λ, μ, ν are indeterminate multipliers. Substituting these values of A_i in (a), we obtain

$$2\sum_{i=1}^{i=n}(\lambda\alpha_i+\mu\beta_i+\nu\gamma_i)\,\alpha_i = \sum_{i=1}^{i=n}\alpha_i^2+C_0$$

$$2\sum_{i=1}^{i=n}(\lambda\alpha_i+\mu\beta_i+\nu\gamma_i)\,\beta_i = \sum_{i=1}^{i=n}\beta_i^2+C_0$$

$$2\sum_{i=1}^{i=n}(\lambda\alpha_i+\mu\beta_i+\nu\gamma_i)\,\gamma_i = \sum_{i=1}^{i=n}\gamma_i^2+C_0$$

These equations may again be written as :—

$$2\lambda\sum_{i=1}^{i=n}\alpha_i^2+2\mu\sum_{i=1}^{i=n}\alpha_i\beta_i+2\nu\sum_{i=1}^{i=n}\alpha_i\gamma_i=\sum_{i=1}^{i=n}\alpha_i^2+C_0 \quad \ldots \quad (1)$$

$$2\lambda\sum_{i=1}^{i=n}\alpha_i\beta_i+2\mu\sum_{i=1}^{i=n}\beta_i^2+2\nu\sum_{i=1}^{i=n}\beta_i\gamma_i=\sum_{i=1}^{i=n}\beta_i^2+C_0 \quad \ldots \quad (2)$$

$$2\lambda\sum_{i=1}^{i=n}\alpha_i\gamma_i+2\mu\sum_{i=1}^{i=n}\beta_i\gamma_i+2\nu\sum_{i=1}^{i=n}\gamma_i^2=\sum_{i=1}^{i=n}\gamma_i^2+C_0 \quad \ldots \quad (3)$$

Subtracting (2) from (1) we obtain

$$2\lambda\left[\sum_{i=1}^{i=n}\alpha_i^2-\sum_{i=1}^{i=n}\alpha_i\beta_i\right]+2\mu\left[\sum_{i=1}^{i=n}\alpha_i\beta_i-\sum_{i=1}^{i=n}\beta_i^2\right]$$

$$+2\nu\left[\sum_{i=1}^{i=n}\alpha_i\gamma_i-\sum_{i=1}^{i=n}\beta_i\gamma_i\right]$$

$$=\sum_{i=1}^{i=n}\alpha_i^2-\sum_{i=1}^{i=n}\beta_i^2$$

$$=(\lambda+\mu+\nu)\left[\sum_{i=1}^{i=n}\alpha_i^2-\sum_{i=1}^{i=n}\beta_i^2\right]$$

$$\text{or } \lambda\sum_{i=1}^{i=n}(\alpha_i-\beta_i)^2-\mu\sum_{i=1}^{i=n}(\alpha_i-\beta_i)^2+\nu\left\{\sum_{i=1}^{i=n}(\beta_i-\gamma_i)^2\right.$$

$$\left.-\sum_{i=1}^{i=n}(\gamma_i-\alpha_i)^2\right\}=0 \quad \ldots \quad (4)$$

$$\left.\begin{aligned} \text{Putting } a^2 &= \sum_{i=1}^{i=n} (\beta_i - \gamma_i)^2 \\ b^2 &= \sum_{i=1}^{i=n} (\gamma_i - \alpha_i)^2 \\ c^2 &= \sum_{i=1}^{i=n} (\alpha_i - \beta_i)^2 \end{aligned}\right\}$$

the equation (4) may be written as

$$\lambda c^2 - \mu c^2 + \nu(a^2 - b^2) = 0$$

$$\text{Similarly,} -\lambda b^2 + \mu(c^2 - a^2) + \nu b^2 = 0$$

Solving these two equations for λ, μ, ν, we obtain

$$\frac{\lambda}{a^2(a^2 - b^2 - c^2)} = \frac{\mu}{b^2(b^2 - c^2 - a^2)} = \frac{\nu}{c^2(c^2 - a^2 - b^2)}$$

$$= \frac{\lambda + \mu + \nu}{a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2}$$

$$= \frac{1}{a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2}$$

$$= \frac{-1}{\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}}$$

Hence

$$A_i = -(\lambda\alpha_i + \mu\beta_i + \nu\alpha_i)$$

$$= \begin{vmatrix} 0 & \alpha_i & \beta_i & \gamma_i \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} \div \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}$$

$$(i = 1,2,3 \ldots n)$$

Thus the co-ordinates of the centre are determined.

§44. To find the co-ordinates of the centre, and also the radius, of a 3-spheric passing through four given points in an n-space.

The method of finding the co-ordinates of the centre is exactly similar to that used in the preceding article. In this case, the co-ordinates are expressed as a linear function of the co-ordinates of the four given points.

To find the radius we proceed as follows :—

Let A, B, C, D, be the four given points and denote them by the numerals 1, 2, 3, 4 respectively.

Let O be the centre of the 3-spheric and denote it by 5.

Let $(1, 2)^2$ stand for the square of the distance between the points 1 and 2, and so on.

Then, since O is the centre, we have

$$R^2=OA^2=OB^2=OC^2=OD^2 \text{ i.e.}$$

$$(1, 5)^2=(2, 5)^2=(3, 5)^2=(4, 5)^2=R^2.$$

The centre of the 3-spheric must lie in the 3-space determined by the 4 given points.

The condition that O lies in the 3-space is given by § 15, as :—

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & (1, 2)^2 & (1, 3)^2 & (1, 4)^2 & (1, 5)^2 \\ 1 & (2, 1)^2 & 0 & (2, 3)^2 & (2, 4)^2 & (2, 5)^2 \\ 1 & (3, 1)^2 & (3, 2)^2 & 0 & (3, 4)^2 & (3, 5)^2 \\ 1 & (4, 1)^2 & (4, 2)^2 & (4, 3)^2 & 0 & (4, 5)^2 \\ 1 & (5, 1)^2 & (5, 2)^2 & (5, 3)^2 & (5, 4)^2 & 0 \end{vmatrix}=0$$

If we put $(1,2)^2=c^2$, $(2, 3)^2,=b^2$, $(2, 3)^2=a^2$,

$(1, 4)^2=f^2$, $(2, 4)^2=g^2$, $(3, 4)^2=h,^2$

The condition reduces to

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & f^2 & R^2 \\ 1 & c^2 & 0 & a^2 & g^2 & R^2 \\ 1 & b^2 & a^2 & 0 & h^2 & R^2 \\ 1 & f^2 & g^2 & h^2 & 0 & R^2 \\ 1 & R^2 & R^2 & R^2 & R^2 & 0 \end{vmatrix} = 0$$

This, when simplified, gives us

$$^{*}2R^2 \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & f^2 \\ 1 & c^2 & 0 & a^2 & g^2 \\ 1 & b^2 & a^2 & 0 & h^2 \\ 1 & f^2 & g^2 & h^2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & c^2 & b^2 & f^2 \\ c^2 & 0 & a^2 & g^2 \\ b^2 & a^2 & 0 & h^2 \\ f^2 & g^2 & h^2 & 0 \end{vmatrix} = 0$$

This gives the radius as the quotient of two determinants.

§45. To find the equations of an r-spheric passing through (r+1) given points, and to find the co-ordinates of its centre and also its radius.

Let the points be denoted by the numerals 1, 2, 3, ... $r+1$; and let their coordinates be denoted by $a_i^{(j)}$ ($j=1, 2, 3, \ldots r+1$; $i=1, 2, 3, \ldots n$).

The equations of any n-spheric may be written as—

$$\sum_{i=1}^{i=n} x_i^2 + 2\sum_{i=1}^{i=n} A_i x_i + C_0 = 0 \quad \ldots \quad \ldots \quad (1)$$

where A_i and C_0 are arbitrary constants.

If this passes through the $(r+1)$ given points, we have the following relations connecting the arbitrary constants $A_1, A_2, \ldots A_n$ and C_0 :—

$$\sum_{i=1}^{i=n} \left(a_i^{(j)} \right)^2 + 2 \sum_{i=1}^{i=n} A_i \left(a_i^{(j)} \right) + C_0 = 0 \quad \ldots \quad (2)$$

$$(j=1, 2, 3, \ldots r+1)$$

---

* *Vide*—Frost, Solid Geometry §589.

Eliminating $A_1, A_2, \ldots A_n$ and C_0 from these equations (1) and (2), we obtain the equations of a set of n-spherics, having a common r-spheric of intersection through the $(r+1)$ given points—

∴ We may take these as the equations of an r-spheric :—

$$\begin{vmatrix} \sum\limits_{i=1}^{i=n} x_i^2 & x_1 & x_2 & \ldots & \ldots & x_n & 1 \\ \sum\limits_{i=1}^{i=n} \left(a_i^{(1)} \right)^2 & a_1^{(1)} & a_2^{(1)} & \ldots & \ldots & a_n^{(1)} & 1 \\ \sum\limits_{i=1}^{i=n} \left(a_i^{(2)} \right)^2 & a_1^{(2)} & a_2^{(2)} & \ldots & \ldots & a_n^{(2)} & 1 \\ \ldots & \ldots & \ldots & & & \ldots & \\ \sum\limits_{i=1}^{i=n} \left(a_i^{(r+1)} \right)^2 & a_1^{(r+1)} & a_2^{(r+1)} & \ldots & \ldots & a_n^{(r+1)} & 1 \end{vmatrix} = 0 \quad \ldots (3)$$

It is possible to choose $A_1, A_2, \ldots A_n, C_0$, so that the centre of the n-spheric lies in the r-space determined by the given r-points *i.e.* in

$$\begin{vmatrix} x_1 & x_2 & x_3 & \ldots & \ldots & x_n & 1 \\ a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & \ldots & \ldots & a_n^{(1)} & 1 \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & \ldots & \ldots & a_n^{(2)} & 1 \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} & \ldots & \ldots & a_n^{(3)} & 1 \\ \ldots & \ldots & \ldots & & & \ldots & \\ a_1^{(r+1)} & a_2^{(r+1)} & a_3^{(r+1)} & \ldots & \ldots & a_n^{(r+1)} & 1 \end{vmatrix} = 0 \quad \ldots (4)$$

In this case the centre of the n-spheric coincides with that of the r-spheric.

To find the coordinates of the centre we solve equations (4) by the method of §§ 43, 44.

We obtain the following equations for determining the coordinates:—

$$A_i = \begin{array}{c} \\ \left|\begin{array}{ccccc} & (1) & (2) & & (r+1) \\ 0 & a_i & a_i & \dots & a_i \\ 1 & 0 & (1,2)^2 & \dots & (1,r+1)^2 \\ 1 & (2,1)^2 & 0 & \dots & (2,r+1)^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & (r+1,1)^2 & (r+1,2)^2 & & 0 \end{array}\right| \end{array} \div$$

$$\left|\begin{array}{ccccc} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & (1,2)^2 & \dots & (1,r+1)^2 \\ 1 & (2,1)^2 & 0 & \dots & (2,r+1)^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & (r+1,1)^2 & (r+1,2)^2 & \dots & 0 \end{array}\right|$$

Let O be the centre and denote it by the numeral $(r+2)$. Then, following the method of § 43, we have

$$\left|\begin{array}{cccccc} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & (1,2)^2 & \dots & (1,r+1)^2 & R^2 \\ 1 & (2,1)^2 & 0 & \dots & (2,r+1)^2 & R^2 \\ 1 & (3,1)^2 & (3,2)^2 & \dots & (3,r+1)^2 & R^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & R^3 & R^2 & \dots & R^2 & 0 \end{array}\right| = 0$$

Simplifying this we obtain

$$2R^2 \left|\begin{array}{ccccc} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & (1,2)^2 & \dots & (1,r+1)^2 \\ 1 & (2,1)^2 & 0 & \dots & (2,r+1)^2 \\ 1 & (3,1)^2 & (3,2)^2 & \dots & (3,r+1)^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & (r+1,1)^2 & (r+1,2)^2 & \dots & 0 \end{array}\right| +$$

$$+\begin{vmatrix} 0 & (1,2)^2 & \dots & (1,r+1)^2 \\ (2,1)^2 & 0 & \dots & (2,r+1)^2 \\ (3,1)^2 & (3,2)^2 & \dots & (3,r+1)^2 \\ \dots & \dots & & \dots \\ (r+1,1)^2 & (r+1,2)^2 & \dots & 0 \end{vmatrix} = 0$$

This determines the radius of the r-spheric through the $(r+1)$ given points.

§46. To find the "content" of an r-spheric in an n-space.

Let a be the radius of the spheric. The "content" is the rth power of the radius multiplied by a constant, *i.e*, it is given by $K_r\, a^r$, where K_r is some constant.

or symbollically, if V_r represent the "content," we have

$$V_r = K_r\, a^r. \quad \dots \quad \dots \quad (1)$$

Take any section of the spheric by an $(r-1)$-space perpendicular to the rth axis, at a distance x from the centre.

Since the section is an $(r-1)$-spheric,* its "content" may be taken to be $K_{r-1}\, y^{r-1}$, where y is the radius of the $(r-1)$-spheric.

Then, $V_r = \int_{-a}^{+a} K_{r-1}\, y^{r-1}\, dx$, where y is the radius of the section. But $x_1^2 + x_2^2 + \dots + x_r^2 = a^2$; $\therefore\ y^2 = a^2 - x^2$.

$$\therefore V_r = K_{r-1} \int_{-a}^{+a} (a^2 - x^2)^{\frac{r-1}{2}}\, dx$$

$$= 2K_{r-1} \int_0^{\frac{\pi}{2}} a^r \cos^r\theta\, d\theta, \text{ putting } x = a\sin\theta,$$

$$\therefore K_r a^r = 2\ K_{r-1}\, .a^r \int_0^{\frac{\pi}{2}} \cos{}^r\theta\, d\theta$$

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[*] Veronese—loc. cit. § 174, Satz III.

$$\therefore\ K_r/K_{r-1} = 2\int_0^{\frac{\pi}{2}} \cos^r\theta\, d\theta.$$

$$= 2\cdot\frac{r-1}{r}\cdot\frac{r-3}{r-2}\cdot\frac{r-5}{r-4} \quad \ldots \quad \frac{4}{5}\cdot\frac{2}{3} \text{ (if } r \text{ is odd)}$$

$$\text{or } = 2\cdot\frac{r-1}{r}\cdot\frac{r-3}{r-2}\cdot\frac{r-5}{r-4} \quad \ldots \quad \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \text{ (if } r \text{ is even)}$$

Putting $r = 2, 3, 4, \ldots$ in succession, we get

$$K_2/K_1 = 2\cdot\tfrac{1}{2}\cdot\tfrac{\pi}{2}$$
$$K_3/K_2 = 2\cdot\tfrac{2}{3}$$
$$K_4/K_3 = 2\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\cdot\tfrac{\pi}{2}$$
$$K_5/K_4 = 2\cdot\tfrac{4}{5}\cdot\tfrac{2}{3}$$
$$\ldots \qquad \ldots$$

Now, K_2 is evidently π. $\therefore K_1 = 2$.

$\therefore$ Multiplying the above factors on both sides respectively we obtain—

$$K_r = 2^r\cdot(\tfrac{1}{2})^{\frac{r}{2}}\cdot(\tfrac{2}{3})^{\frac{r-2}{2}}(\tfrac{3}{4})^{\frac{r-2}{2}}(\tfrac{4}{5})^{\frac{r-4}{2}}(\tfrac{5}{6})^{\frac{r-4}{2}} \quad \ldots$$

$$\ldots \left(\frac{r-4}{r-2}\right)^2\left(\frac{r-3}{r-2}\right)^2\left(\frac{r-2}{r-1}\right)\left(\frac{r-1}{r}\right)\left(\frac{\pi}{2}\right)^{\frac{r}{2}}$$

when r is even ... ... (A)

$$\text{or } = 2^r\cdot(\tfrac{1}{2})^{\frac{r-1}{2}}(\tfrac{2}{3})^{\frac{r-1}{2}}(\tfrac{3}{4})^{\frac{r-3}{2}}(\tfrac{4}{5})^{\frac{r-3}{2}}(\tfrac{5}{6})^{\frac{r-5}{2}}(\tfrac{6}{7})^{\frac{r-5}{2}} \quad \ldots$$

$$\left(\frac{r-4}{r-3}\right)^2\left(\frac{r-3}{r-2}\right)^2\left(\frac{r-2}{r-1}\right)\left(\frac{r-1}{r}\right)\left(\frac{\pi}{2}\right)^{\frac{r-1}{2}}$$

when r is odd ... ... (B)

From the formulæ, (A) and (B) we may calculate the "content" of spherics of any number of dimensions up to n.

Thus, putting $r=2$, from (A) we obtain

$$K_2 = 2^2\cdot(\tfrac{1}{2})\cdot\tfrac{\pi}{2} = \pi$$

$\therefore$ V_2 = area of a circle $= \pi a^2$, a being the radius.

Putting $r=3$, from (B) we have

$$K_3 = 2^3 \cdot (\tfrac{1}{2}) \cdot \tfrac{2}{3} \cdot \tfrac{\pi}{2} = \tfrac{4}{3}\pi,$$

$\therefore$ V_3 = Volume of a sphere $= \frac{4}{3}\pi a^3$.

Putting $r=4$, from (A) we get

$$K_4 = 2^4 \cdot (\tfrac{1}{2})^2 (\tfrac{2}{3}) (\tfrac{3}{4}) (\tfrac{\pi}{2})^2$$

$$= \pi^2/2$$

$\therefore$ V_4 = content of the Hyper-sphere $= \frac{1}{2}\pi^2 a^4$.*

Note.—It is interesting to note that as we pass on to spherics of dimensions higher than three, the expression for the "content" contains higher powers of 'π.' It is further noticed that the power of π in the expression is increased by unity for every second higher dimensions. Thus for spaces of 4 and 5 dimensions the power is 2, for 6 and 7 dimensions it is 3, and so on. Also, for 2 and 3 dimensions the power is unity.

§47. To find the "Surface Content" of an r-spheric.

The surface of an r-spheric is of $(r-1)$ dimensions.

Hence we may take the "Surface-content" to be equal to the $(r-1)$th power of the radius multiplied by a certain constant K_r, *i.e.*, if S_r represents the surface-content, we have

$$S_r = K_r . a^{r-1}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

Take a section of the spheric by an $(r-1)$-space at right angles to the rth axis.

The section will be an $(r$-$1)$-spheric. Now consider the surface-content of a section of the r-spheric of very small extension dx_r in the direction of the rth axis.

Let ds_r represent the length of the arc of the strip in this direction.

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* Vide—H. P. Manning, Geometry of four dimensions, § 165, Cor I. Manning calls this "content" of the Hyper-sphere—"The Hyper-volume."

Then, $S_r = \int K_{r-1} y^{(r-2)} ds_r$ where y_r is the radius of the $(r-1)$-spheric and $y_r^2 = a^2 - x_r^2$.

But $ds_r^2 = dx_1^2 + dx_2^2 + \dots + dx_{r-1}^2 + dx_r^2 = dx_r^2 + dy_r^2$

$$\therefore\ ds_r = \frac{ds_r}{dx_r}.\,dx_r = \sqrt{1 + \left(\frac{dy_r}{dx_r}\right)^2}\, dx_r = \frac{a}{y_r}\, dx_r$$

$$\therefore\ S_r = \int_{-a}^{+a} K_{r-1}\, y_r^{(r-2)}\, \frac{a}{y_r}\, dx_r$$

$$= \int_{-a}^{+a} K_{r-1}\, y_r^{(r-3)}\, a.\, dx_r$$

$$= 2\, K_{r-1} \cdot a \int_0^a \left(a^2 - x_r^2 \right)^{\frac{r-3}{2}} dx_r$$

$$= 2\, K_{r-1}\, a^{r-1} \int_0^{\frac{\pi}{2}} \cos^{(r-2)} \theta\, d\theta \text{ (putting } x_r = a \sin \theta)$$

$$\text{or, } K_r . a^{r-1} = 2\, K_{r-1}\, a^{r-1} \int_0^{\frac{\pi}{2}} \cos^{(r-2)} \theta\, d\theta$$

$$\therefore\ K_r/K_{r-1} = 2 \int_0^{\frac{\pi}{2}} \cos^{(r-2)} \theta\, d\theta$$

$$= 2 \cdot \frac{r-3}{r-2} \cdot \frac{r-5}{r-4} \cdots \frac{2}{3} \text{ (when } r \text{ is odd)}$$

$$\text{or } = 2 \cdot \frac{r-3}{r-2}\ \frac{r-5}{r-4} \cdots \frac{3}{4}\ \frac{1}{2}\ \frac{\pi}{2} \text{ (when } r \text{ is even).}$$

Putting $r=4, 5, 6, \ldots\ldots$ in succession we get

$K_4/K_3 = 2. \frac{1}{2} \cdot \frac{\pi}{2}$

$K_5/K_4 = 2. \frac{2}{3}$.

$$K_6/K_5=2.\frac{1}{2}.\frac{3}{4}.\frac{\pi}{2}.$$

... ...

$$K_r/K_{r-1}=2.\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\dots\quad\dots\frac{r-3}{r-2}.\frac{\pi}{2}.\ (r\text{ even}).$$

$$\text{or}=2.\frac{2}{3}.\frac{4}{5}.\frac{6}{7}.\dots\quad\dots\frac{r-3}{r-2}.\ (r\text{ odd}).$$

But $K_3=4\pi=2^3.\frac{1}{2}\pi=2^3.\ \frac{\pi}{2}.$

∴ Multiplying the factors on both sides respectively we get

$$K_r=2^r\ \left(\frac{1}{2}\right)^{\frac{r-2}{2}}\left(\frac{2}{3}\right)^{\frac{r-4}{2}}\left(\frac{3}{4}\right)^{\frac{r-4}{2}}\left(\frac{4}{5}\right)^{\frac{r-6}{2}}\left(\frac{5}{6}\right)^{\frac{r-6}{2}}\quad\dots\quad\dots$$

$$\dots\left(\frac{r-6}{r-b}\right)^2\left(\frac{r-5}{r-4}\right)^2\left(\frac{r-4}{r-3}\right)\left(\frac{r-3}{r-2}\right)\left(\frac{\pi}{2}\right)^{\frac{r}{2}}\quad\dots\quad(A)$$

(when r is even)

$$\text{or}=2^r.\ \left(\frac{1}{2}\right)^{\frac{r-3}{2}}\left(\frac{2}{3}\right)^{\frac{r-3}{2}}\left(\frac{3}{4}\right)^{\frac{r-5}{2}}\left(\frac{4}{5}\right)^{\frac{r-5}{2}}\dots\quad\dots\quad\dots$$

$$\dots\left(\frac{r-6}{r-5}\right)^2\left(\frac{r-5}{r-4}\right)^2\left(\frac{r-4}{r-3}\right)\left(\frac{r-3}{r-2}\right)\left(\frac{\pi}{2}\right)^{\frac{r-1}{2}}\quad\dots\quad(B)$$

(when r is odd)

Therefore by means of formulae (1), (A) and (B) we may calculate the "surface-content" of spherics of any number of dimensions.

Thus, when $r=2$, from (A) we get

$$K_2=2^2.\ \frac{\pi}{2}\ =2\pi.$$

∴ S_2 = circumference of a circle $=2\pi a$.

When $r=3$, from (B) we get

$$K_3=2^3.\frac{\pi}{2}=$$

$\therefore$ S_3 = Surface of the sphere $=4\pi a^2$.

When $r=4$, from (A) we get

$$K_4=2^4.\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)^2=2\pi^2.$$

$\therefore$ S_4 = "Surface-content" of the Hypersphere $=2\pi^2 a^3$*

Note.—Here also we find that the expression for the "surface-content" contains higher powers of π when we pass on to spherics of dimensions higher than three, and the power increases by unity for every alternate dimensions.

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* *Cf.* Manning—Geometry of four dimensions §156. He names this "surface-content" as the volume of the Hyper-sphere.